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ANALYSIS OF CONCRETE SLABS ON GROUND AND SUBJECTED TO WARPING AND MOVING LOADS

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BY

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Technical Paper

ANALYSIS OF CONCRETE SLABS ON GROUND AND SUBJECTED TO WARPING AND MOVING LOADS

To: J. F. McLaughlin, Director
Joint Highway Research Project

March 21, 1969

File: 9-7-4

From: H. L. Michael, Associate Director
Joint Highway Research Project

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The Technical Paper attached is submitted for approval of publication by the Highway Research Board. The paper "Analysis of Concrete Slabs on Ground and Subjected to Warping and Moving Loads" by K. H. Lewis and M. E. Harr was presented at the 1969 Annual Meeting of the Highway Research Board.

The paper is a summary of a research report presented to the JHRP Board at an earlier date. The title of that Report was the same as the title of the attached Technical Paper.

The paper is submitted to the Board for action.

Respectfully submitted,

Harold L. Michael

Harold L. Michael
Associate Director

HLM:rg
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Technical Paper

ANALYSIS OF CONCRETE SLABS ON GROUND AND SUBJECTED TO WARPING AND MOVING LOADS

by

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Joint Highway Research Project
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Purdue University
Lafayette, Indiana
March 21, 1969

ANALYSIS OF CONCRETE SLABS ON GROUND AND SUBJECTED TO WARPING AND MOVING LOADS

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ABSTRACT

A theory has been developed whereby stresses and deflections could be calculated for a series of rectangular slabs lying on a viscoelastic foundation and subjected to a moving load. The stresses and deflections are caused by the weight of the slab, the moving concentrated load, and by linear temperature (or moisture)variations that cause sufficient warping so that the slab is only partially supported by its foundation.

Part I considers the problem of partial support caused by warping (in this study, the temperature at the top of the slab is smaller than that at the bottom), while Part II concerns itself with the effect of a reduction in subgrade support over some narrow region.

In both parts, the support conditions were simulated by a Kelvin visco-elastic model, and zones (which depended on the value of subgrade reaction) were set up so that the solutions to the governing differential equations could be reduced to a set of simultaneous algebraic equations. For the problem studied in Part I, the resulting simultaneous equations were non-linear and a method of functional iteration equivalent to an N-dimensional Newton's Method was used. In the case of Part II, the equations were linear and the Method of Crout Reduction was used. In both parts, the equations were solved with the aid of an IBM 7090 digital computer using a Fortran source program.

It was found that when partial support due to warping exists, the tensile stress in the slab can increase with increasing velocity of load. Moreover, the maximum deflection (downward) need not occur when the velocity of the load is equal to zero. The reduction in subgrade support over a narrow region (approximately 8 feet or less) does lead to deflections and stresses which are higher than those calculated using the initial value of subgrade reaction. This is particularly evident when the load is over the region of reduced subgrade reaction.

INTRODUCTION

Improvements in the performance of concrete pavements for highways and airports have for many years been the cause of concern among Civil Engineers interested in these problems. This concern has not only been due to the ever increasing cost of construction and maintenance, but also to the preponderance of cracks which exist in pavements.

The development of these cracks depends on several factors, (such as type of subgrade, deterioration of the concrete, temperature effects and load) (52), and, although they may or may not represent a failed condition, they do

indicate deficiencies in the analysis, design or construction of highway pavements. Cracks not only detract from the general appearance of a highway, but often they lead to driving discomfort and pavement deterioration; especially when load transfer and subgrade support are lost.

As far as the construction of concrete pavements is concerned, advancement in the development of the principles of soil mechanics has made great strides in limiting the use of poor subgrades and has virtually eliminated the use of soils susceptible to frost action and pumping. In addition, specifications (1, 47) geared at promoting the use of sound concrete and the practice of good construction methods, have become widespread.

Along with refinements in construction procedures, methods of monitoring the factors influencing concrete pavements have been developed (15, 16), and strains as well as deflections of pavements can be determined for both static and dynamic loadings (11, 18). Temperature differences between the upper and lower faces of the slab can be measured with the use of thermo-couples (51), and variations in moisture content can be established from the dielectric properties of concrete (3).

Today, the problem of material weakness has been reduced, if not solved, and significant progress has been made in the ability to build pavements and to measure their properties which seem important; however, only little has been done to upgrade the methods of analysis. Current procedures for designing and evaluating pavements (52) are still based on static loads and except for introducing equivalent static loadings (2), they do not account for the dynamic response of the pavement to moving loads. Moreover, the pavement is usually assumed to be fully supported at all times even though this often is not the case (11, 21, 22).

The present trend in transportation is towards heavier loads and higher speeds and consequently there is an increasing need to be able to predict performance. This can only be done when more is known about the input and causal factors of traffic loads, curling, and the loss of subgrade support. It is to help satisfy this need, that this paper is dedicated.

BACKGROUND

In the early stages of development, pavement design and road construction consisted of a collection of rule of thumb procedures based on empirical knowledge. As early as 1901, test roads were being used to determine the best type of pavement for the prevailing traffic (38). Between 1920 and 1940, much work was done on the classification of soils (5) and Highway Engineers were able to correlate pavement performance with subgrade types.

With the advent of World War II, engineers were faced with the problem of designing pavements for greater wheel loads than previously thought necessary. They realized that they could not afford the long period of time which was required to develop design procedures based on past experiences and thus sought a more rational basis of design. This search ultimately led to procedures which form the core of pavement design today.

Static Load Solutions

In 1884, Hertz (13) first published a solution to the problem of an elastic plate on a Winkler-type foundation (50); however, it was not until the advent of Westergaard's solution in 1926 that a highway pavement was treated in this manner and the problem was approached from a purely theoretical point of view. Today, this work still forms the basis of the analytical bent in pavement design.

Westergaard (48) solved the problem of a slab fully supported on a Winkler foundation and subjected to static loads applied at the interior, free edges and corners of the slab. Later, Kelley (23), Spangler (43), Pickett (34) and Westergaard (49) himself extended these original solutions to account for linear temperature variations. In 1957, Freudenthal and Lorsch (10) used the three fundamental models (Maxwell, Kelvin and Standard Solid) to study the problem of an infinite beam on a viscoelastic foundation; and in 1958, Hoskin and Lee (19) solved the problem of an infinite plate on a viscoelastic foundation.

The foregoing solutions neglect the shearing forces generated at the pavement-base interface. Several mechanisms have been offered to account for this effect. Filonenko- Borodich (9) considered a set of springs held together by a membrane, while Schiel (42) took a fluid which exhibited surface tension as his soil model. Pasternak (32) and Kerr (25) on the other hand, considered a beam of unit depth resting on a bed of interrelated springs as their foundation, and Pister and Williams (33) used the shear interactions suggested by Reissner (40). Perhaps the most realistic approach offered to account for shear in an elastic base is that given by Klubin (26) who expressed the pavement reaction by an infinite series of Tschebyscheff polynomials which also accounts for the two elastic constants (Modulus of Elasticity and Poisson's Ratio).

It should be noted that Klubin's solution is in fact an elastic solution while the others, which impose Winkler assumptions, are not. In spite of this, since the Winkler foundation affords a much simpler analysis and generally gives good agreement with field data, it will no doubt remain popular.

All of the above analyses are based on the assumption that the slab maintains contact with its support at all points and at all times. Experimental and field studies (16, 21, 22) have shown this assumption to be seriously in error and a few investigators have accounted for the effects of only partial support. In 1959, Leonards and Harr (28) solved the problem of a partially supported slab on a Winkler foundation for linear temperature and/or moisture variations. Later, Reddy, Leonards, and Harr (39) extended this analysis by introducing non-linear temperature variations as well as a viscoelastic foundation. In all of these analytical procedures, only the case of symmetrical, statically applied loads were considered.

Dynamic Load Solutions

For many years, the problem of determining the stresses and deflections in a vibrating plate has been of interest primarily to mathematicians. Raleigh (37) and Lamb (27) using the classical beam theory developed by Euler (7) and Bernoulli (4), studied several problems dealing with the vibration of bars, membranes and plates. Later, Ritz (41) elaborated on this work and made significant contributions toward the study of vibrating rectangular plates.

Recently, engineers have felt the need to account for the dynamic response of pavements and several solutions have evolved. Pioneering these solutions was the work of Timoshenko (45), Hovey (20) and Ludwig (30) in their studies of the dynamics of rails subjected to moving loads. In 1943, Dorr (6) using Fourier integrals extended the idea to a beam, but in all of these solutions the foundation was represented by a Winkler model which exhibited no viscous effects.

Later, in 1953, Kenney (24) added the effects of linear damping and by means of the method of undetermined coefficients was able to examine the relationship between deflections and critical velocity. In a more recent work, Thompson (44) elaborated on Kenney's solution and showed that the solutions for the deflections fall into three distinct regimes. Specifically, Thompson showed that these regimes were defined by the value of the discriminant of the fourth order characteristic equation. If the value of the discriminant as defined by

$$\Delta = 16\,[4\,(1-\epsilon^2)\,\theta^8 - (8 - 36\epsilon^2 + 27\epsilon^4)\,\theta^4 + 4] \qquad (1)$$

where ϵ = the damping ratio

and θ = the velocity ratio

is greater than zero, the characteristic equation has no real roots. If Δ is equal to zero, there is one, real, double root; and if Δ is less than zero there are two, real, unequal roots.

The solutions presented by Thompson demonstrate that at static conditions, the deflection curve is symmetrical (with maximum deflection occurring under the load); but as velocity increases, the point of maximum deflection falls further and further behind the load. Upward deflections occur behind the moving load when Δ is greater than zero, however when Δ is less than zero, there are no upward deflections (i.e. the deflected surface does not intersect the axis of zero deflection, but simply approaches it at some great distance behind the load).

Several investigators have also considered the road loading system as an interaction between two major components which are interdependent, namely vehicles and roads. Fabian, Clark and Hutchinson (8) examined the elements of each component and developed some basic mathematical models. Analysis of their vehicle sub-system showed that the magnitude of the dynamic load is

a function of vehicle dynamic properties and apparent road profile, and may in fact be significantly greater than the static load. Quinn and De Vries (35) used an experimental procedure to determine the highway and vehicle characteristics and showed that if these quantities are known, the reaction of a vehicle on a highway can be predicted.

In general, most of the analytical studies reported in the literature on moving loads on pavements (52), do not account for the interaction between vehicle and road. For the most part, these reports show that for fully supported pavements, the lower the velocity, the greater the deflections and stresses in the slab. Measurements of displacement (15, 17, 18) generally support this finding; however, some data (11) does exist to suggest the possibility that displacements may increase at velocities greater than 30 to 40 mph. Also to be noted is the fact that tests on tire forces on pavements (36) do indicate that the "average" force produced on the pavement by a moving load, increases with velocity.

Obviously, there are several interdependent factors such as the vehicle's suspension system, tire pressures and road profile which should be included in the analysis of pavements; however at the present time, the interaction between vehicle and pavement is not well understood. In spite of this, further insight into the highway problem can be gained by examining some of the simplifying assumptions which appear in current analyses of highway pavements. Thus, it is the object of this paper to obtain by analytical means the stresses and deflections in a partially supported concrete slab when subjected to a moving load. Such partial support may be caused by temperature and moisture gradients or by the weakening and partial or complete loss of subgrade.

FORMULATION OF PROBLEM

In the first part of this paper, the problem is considered of a slab lying on a foundation while subjected to moving loads as well as to temperature (and/or moisture) gradients which would cause upward curling.[1] The second part will investigate conditions where a weakening of the subgrade has developed. The conditions in Part II may exist where (a) water has infiltrated under the pavement from the side of the roadway; or (b) where a joint has opened or where a crack has occurred in a warped pavement to such an extent as to enable it to regain complete contact with the foundation. When such an opening exists, water may infiltrate through the surface of the pavement. Under both conditions (a) and (b), the infiltration may weaken the subgrade and may eventually lead to pumping. The following assumptions are made for both parts in order to render them tractable:

ASSUMPTIONS

1. A highway pavement can be represented by an array of rectangular plates.
2. The usual assumptions of plate theory hold, namely: the plate is homogeneous, isotropic and obeys Hooke's law; deflections are small in comparison to thickness; plane cross sections normal to the middle plane in the unstressed condition remain normal to this surface after bending; the effects of rotatory inertia and shear deformation can be neglected.
3. The highway base material acts like a set of linear viscoelastic elements. The inertia of the material is neglected.
4. External forces acting on the plate are those due to a constant-velocity line load and gravity.

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[1] Upward curling is understood to exist when temperature gradients cause the ends of the slab to rise vertically.

5. The plates are subjected to changes in temperature (and/or moisture) which vary linearly with depth. The variation in temperature is constant on all planes parallel to the upper and lower plate surface and is independent of time.

Part I

In addition to the foregoing assumptions, further assumptions regarding the interaction between slabs are required. For the case studied here, it is assumed that no bending moment exists between slabs (this will be discussed later). However, some load transfer can be accounted for by specifying a shearing force between slabs, equivalent to that existing in an infinite slab.

In general, the governing differential equation describing the pavement section illustrated in Figure 1 can be expressed as follows (31):

$$D\left(\frac{\partial^4 w}{\partial x_1^4} + 2\frac{\partial^4 w}{\partial x_1^2 \partial y_1^2} + \frac{\partial^4 w}{\partial y_1^4}\right) + \rho H \frac{\partial^2 w}{\partial t^2} =$$

$$q(x_1, y_1, t) - p(x_1, y_1, t) \qquad (2)$$

where D = flexural rigidity of the slab

w = mid-plane deflection of the slab (positive downward)

x_1, y_1 = fixed coordinates

ρ = density of the slab

H = slab thickness

q = surface loading

p = foundation reaction

t = time

Using Assumption 3, the foundation reaction, p, may be expressed as (see Figure 1b): $p\ (x_1, y_1, t) = C \frac{\partial w}{\partial t} + Kw$, where C is the damping coefficient and K is the modulus of subgrade reaction. Applying the loading conditions implied by Assumption 4, and assuming that the deflection does not vary in the y_1 direction, Equation 2, for a constant cross section of pavement, becomes

$$D \frac{\partial^4 w}{\partial x_1^4} + \rho H \frac{\partial^2 w}{\partial t^2} + C \frac{\partial w}{\partial t} + Kw = q_o + P(x_1, t) \qquad (3)$$

in which q_o = unit weight of slab

and P = the moving line load

Employing the transformation, $x = (x_1 - vt)$, the equation is seen to reduce to a function of only the one variable, x. Now if in addition the moving load, P, is introduced with the boundary conditions, the following differential equations are obtained:

For zones 1 and 4 $$D \frac{d^4 w}{dx^4} + \rho H v^2 \frac{d^2 w}{dx^2} = q_o \qquad (4a)$$

For zones 2 and 3 $$D \frac{d^4 w}{dx^4} + \rho H v^2 \frac{d^2 w}{dx^2} - Cv \frac{dw}{dx} + Kw = q_o \qquad (4b)$$

The boundary conditions for the indicated zones may be summarized as follows:

$$\begin{aligned} &(a) \quad M_1(-b) = 0 \\ &(b) \quad V_1(-b) = \bar{V}(-b) \\ &(c) \quad w_1(-c) = 0 \\ &(d) \quad w_2(-c) = 0 \end{aligned} \qquad (5)$$

(e) $w_1'(-c) = w_2'(-c)$

(f) $w_1''(-c) = w_2''(-c)$

(g) $w_1'''(-c) = w_2'''(-c)$

(h) $w_2(0) = w_3(0)$

(i) $w_2'(0) = w_3'(0)$

(j) $w_2''(0) = w_3''(0)$

(k) $w_3'''(0) - w_2'''(0) = P/D$

(l) $w_3(d) = 0$

(m) $w_4(d) = 0$

(n) $w_3'(d) = w_4'(d)$

(o) $w_3''(d) = w_4''(d)$

(p) $w_3'''(d) = w_4'''(d)$

(q) $M_4(a-b) = 0$

(r) $V_4(a-b) = \bar{V}(a-b)$

Solution of Problem

To obtain the solution for the case where the warped slab is as represented by Figure 1, the value of Δ (Equation 1) is determined and Equations 4a and 4b are solved to obtain expressions for the deflections and stresses in the slab (29). The constants in the resulting set of non-linear

equations are then evaluated using the conditions listed in Equation 5. In solving the set of non-linear equations, an initial estimate of the solution was made using the fully supported slab, then a method of functional iteration equivalent to an N-dimensional Newton's Method was used with an IBM 7090 computer. With the constants determined, the stresses and displacements were then obtained. Using this same procedure, the other deflection patterns resulting from the moving load were analyzed (see Figure 2).

Results

Due to the large number of variables involved, it is impractical to present the results for all considered cases. Instead, numerical results are given for a few typical cases to illustrate general trends and for purposes of comparison with results previously obtained by others.

Using the left edge of the slab, as origin, the moving load was considered at intervals of 4 ft. and stresses and deflections were determined for combinations of the following data:

μ = 0.15

E = 4×10^6 psi.

a = 480 ins.

H = 8, 10, 12 ins.

K = 100, 200 pci.

C_r = 1.5, 2.0

v = 0, 20, 40, 60, 80 mph.

P = 125 lbs/in. (The value of P was determined using an axle load of 18,000 lbs. over a pavement 12 ft. wide)

q_o = 150.9 pcf.

ΔT = 20, 30, 40 deg. F

β = 6×10^{-6} inches per inch per deg. F.

Typical curves for deflections and stresses are presented in Figure 3 whereas Figures 4 and 5 show how maximum deflection and maximum stress vary with velocity when the difference between slab surfaces is 20 and 40 degrees Fahrenheit. A record of the movement experienced by the ends of the slab is presented in Figures 6 and 7 as a function of the load's position.

Discussion of Part I[2]

In developing the theory for this part of the analysis, it was assumed that a pavement could be represented by an array of rectangular slabs. Furthermore, it was assumed that the bending movement between slabs could be neglected while a shearing force equivalent to that in an infinite slab could be used to provide shear transfer. This seemed justifiable because of the relatively short depth of dowel embedment and the general nature of expansion and contraction joints are such as to enable the transfer of only very limited bending. For most highway work, dowel bars are only approximately 2 feet long compared to the length of a slab which averages about 40 feet; moreover, they are, generally, smooth and lubricated at one end in order to maintain freedom of horizontal movement between slabs. Under repetitive loading, these joints become looser and conceivably act more like a hinge thus offering little or no moment transfer. On the other hand, substantial shear transfer could be experienced if the joint opening is small and the deflections are large enough to cause the development of a bearing pressure between dowel and concrete.

[2] The plan is to treat each part separately and then summarize them.

In any case, it should be noted that the magnitude of the moment and shear is really only significant in the near vicinity of the load itself. As a result, the solution is seen to be influenced by the values used for moment and shear transfer only when the load approaches the edges of the slab.

Figures 3, 4, and 5 demonstrate that for the case studied here, the damping ratio, C_r, does not greatly influence the values obtained for deflection and stress at low to moderate velocities; however, the higher value of C_r does result in a wider and less deflected trough (i.e. the depression under the load is wider but of smaller amplitude). In general, these figures tend to indicate that the pattern of the deflection and stress curves is mainly determined by temperature difference between slab surfaces and by the position of the moving load.

In the case considered here, temperature differences cause the ends of the slab to curl upwards and become unsupported, while points midway between the mid-point and the ends of the slab experience an increase in positive deflection. For the positions of load shown in Figure 3, there is an increase in positive deflection and a decrease in tensile stress in the near vicinity of the load; however, as Figures 6 and 7 indicate, the radius of influence (wave length of deflected surface) is not only a function of load position but also one of velocity. At low velocities, the radius of influence is small, but as velocity increases, this radius increases significantly behind the moving load. As a matter of fact, this characteristic which was also observed by Thompson, plays an important role in the explanation of Figures 4 and 5.

For the case of an overdamped pavement ($C_r > 1.0$), it was initially anticipated that both the maximum deflection and stress would decrease

with increasing velocity; however, Figures 4 and 5 indicate that if curling is in evidence this may not be the case. As was stated earlier, the maximum positive deflection in an unloaded slab subjected to moisture and/or temperature gradients which cause upward curling at the ends, occurs somewhere near the 1/4 and 3/4 points of the slab. When a moving load is introduced, a wave train is set up and the deflected trough which lags the load becomes wider as velocity increases. In other words, the influence of the deflected trough behind the moving load increases with increasing velocity.

Therefore, if a slab which is initially curled upwards at the ends is subjected to a moving load, there will be a tendency for the portion of the slab behind the load to become flatter and more fully supported as velocity increases, see Figures 6 and 7. As this flattening occurs, the maximum positive (downward) deflection behind the load increases until the velocity reaches a value which produces a reasonably flat, fully supported slab; then, a decrease in maximum positive deflection is experienced with further increases in velocity. Thus in a slab which is curled upwards at the ends the maximum positive deflection does not occur at zero velocity, as is the case for fully supported slabs not subjected to moisture and/or temperature gradients, but as some velocity greater than zero.

As far as stresses are concerned, a line of reasoning similar to that used before to explain the deflection pattern, may be employed to account for the increase in maximum stress with increasing velocity as shown in Figures 4 and 5. Increased velocities result in a greater tendency to flatten the slab and this in turn causes an increase in the tensilestress at the top of the slab.

As may be expected, the pattern as well as the magnitude of deflections and stresses obtained depend on the stiffness of the slab, the firmness of the subgrade material and the temperature difference existing between slab surfaces. For the range of velocity studied, maximum positive deflection decreased as the value of subgrade reaction increased, and increases were experienced as pavement thickness and temperature gradients became greater. However, as Figures 4 and 5 reveal, the greater the resistance to flattening, the higher is the velocity at which the curled surface becomes flat enough to result in a decrease in maximum positive deflection with increasing velocity. In the case of stresses, the increase in temperature gradient from 20 degrees to 40 degrees Fahrenheit produced almost a 50 percent increase in stress, while the variation in pavement thickness and subgrade reaction did not appear to have much influence. The influence of the higher value of C_r on deflection and stress is also small, however the fact that it produces a wider and less deflected trough is quite evident.

Part II

In this part of the paper, the main objective is to determine what effect a reduction of the subgrade reaction would have on the stresses and deflections in a pavement subject to imposed vehicular loadings. For this case there is little loss in generality by assuming a particular value of shear and moment transfer between slabs. The procedure to be followed can be adapted to any degree of transfer, hence for the present purpose wherein there is primary concern only for the region evidencing subgrade reduction, it is expedient to consider primarily the case of an infinite slab. As a special case, the condition that 50% shear and no moment transfer exist between two semi-infinite slabs is also examined.

The pavement section for the case of the infinite slab is illustrated in Figure 8. Here, the subgrade in its original form is represented by zones 1 and 3, while the area over which the reduction of subgrade reaction occurs is represented by zone 2. The differential equation describing the surface of the pavement is given by Equation 4b.

To introduce the equivalence of a moving load, another zone must be added to the pavement, for example in Figure 8, zone 1 is seen to be divided into two zones, 1a and 1b; and the force P is accounted for in the boundary conditions.

Solution of Problem

For the case when the moving load is over zone 1 and approaching zone 2, Equation 4b is applied to each of zones 1a, 1b, 2 and 3, and solutions are obtained using the appropriate subgrade properties. The boundary conditions used in evaluating the constants are as follows:

$$
\begin{aligned}
&(a) \quad w_{1a}(-\infty) = q_o/K_o \\
&(b) \quad w'_{1a}(-\infty) = 0 \\
&(c) \quad w_{1a}(0) = w_{1b}(0) \\
&(d) \quad w'_{1a}(0) = w'_{1b}(0) \\
&(e) \quad w''_{1a}(0) = w''_{1b}(0) \\
&(f) \quad w'''_{1b}(0) - w'''_{1a}(0) = P/D \\
&(g) \quad w_{1b}(L_1) = w_2(L_1) \\
&(h) \quad w'_{1b}(L_1) = w'_2(L_1) \\
&(i) \quad w''_{1b}(L_1) = w''_2(L_1) \\
&(j) \quad w'''_{1b}(L_1) = w'''_2(L_1)
\end{aligned}
\qquad (6)
$$

[illegible]

[illegible]

[illegible]

(k) $w_2(L_2) = w_3(L_2)$

(l) $w_2'(L_2) = w_3'(L_2)$

(m) $w_2''(L_2) = w_3''(L_2)$

(n) $w_2'''(L_2) = w_3'''(L_2)$

(o) $w_3(\infty) = q_o/K_o$

(p) $w_3'(\infty) = 0$

After evaluating the constants in the resulting set of linear equations (29) using the Method of Crout Reduction with an IBM 7090 computer, the deflections and stresses were determined. The complete solution to the problem was obtained by applying this same procedure with suitable modifications to the schemes shown in Figure 8.

Results

To investigate the effect of a reduction in subgrade reaction, numerical results were obtained for combinations of the following data:

μ = 0.15

E = 4×10^6 psi

ℓ = 24, 48, 96 ins.

H = 8, 10, 12 ins.

K_o = 100, 200 pci

K_1 = 0, 25, 50, 75, 100, 150, 200 pci

C_F = 1.5, 2.0

v = 0. 20, 40, 60, 80 mph

P = 125 lbs/in.

q_o = 150.9 pcf

Some typical curves (in non-dimensional form) for displacements and stresses for three positions of the load are presented in Figure 9. Values for the deflection and stress under a static load, P, are given in Table 1. These values may be used in Figure 9 with appropriate amplification ratios, to obtain the magnitudes of deflection and stress at a point.

Table 1

Deflection and Stress Under a Static Load of 125 lbs/in

	K_o = 100 pci		K_o = 200 pci	
H ins	w_o ins	σ_o psi	w_o ins	σ_o ins
8	.019144	150.607	.010722	126.645
10	.019017	113.948	.010481	95.819
12	.019449	90.726	.010573	76.291

Figure 10 illustrates the variation of maximum stress and deflection with velocity, while Figures 11 and 12 reveal the influence of the parameters ℓ and H. Curves similar to those given in Figure 9 are presented in Figure 13 for the special case where there is only 50% load transfer and no moment transfer across a discontinuity in the pavement's surface.

Discussion of Part II

As shown in Figure 9, there is an increase in positive deflection amplification as the value of K_1/K_o decreases. For the load positions shown, the increase is greatest when the load is over the zone of reduced subgrade reaction and least when the load is passed this zone. As may be expected, when the load is moving over a homogeneous subgrade material, the point of maximum positive deflection lags the position of the load; however, if there

is a zone of reduced subgrade reaction and the load is approaching this zone, as is shown in Figure 9a, the maximum deflection can lead the position of the load at low values of K_1/K_o. For the cases when the load is over the weakened region or has passed it, maximum positive deflection lags the load. This characteristic tends to increase as the load moves out of the weakened area.

As regards to stresses, a reduction in the value of K does not appear to influence the maximum stress very much unless the load is within the area of reduced-K; however, as Figure 9 demonstrates, it can effect significantly the shape of the curve at very low values of K_1/K_o.

For the three positions of load considered, the maximum stress, which always occurred under the moving load, reached its greatest value when the load was over the weakened zone; here, stress increased as K_1/K_o decreased. In the case when the load is approaching the weakened zone stress is again seen to increase with decreasing ratio, K_1/K_o, but after the load has passed this zone, the trend is reversed and the lower the value of K_1/K_o, the lower the stress under the load. This means that if a slab is lying on a subgrade which exhibits a soft spot, the stress experienced in the slab is greater when the load is moving towards the center of this weakened zone that when it (the load) is moving away. Consequently, if failure due to overstressing does occur, the signs of distress should appear somewhere between the center and the "back" edge of the soft zone.

Figure 10 demonstrates that except for very low values of K_1/K_o, deflection and stress amplification decreases with increasing velocity, however the decrease appears to be more pronounced for the smaller value of initial subgrade reaction, K_o. Here, as in Part I, the higher value of C_r yields the lower value of maximum deflection and stress.

The influence of the length of weakened region is illustrated in Figures 11a and 11b. In these figures, both the maximum stress and deflection are seen to increase for all values of K_1/K_o less than 1, as the region of reduced subgrade reaction becomes larger. The stiffer the subgrade material is initially, the greater the increase in deflection and stress. However, in the case of stress, this is only discernible at low values of K_1/K_o.

In general, the influence of the thickness of the pavement upon stress and deflection amplification was only significant at low ratios of K_1/K_o. As is shown in Figures 12a and 12b, deflection amplification decreases as thickness increases, when the value of K_1/K_o is approximately equal to 0.7 or less; while stress amplification remains relatively insensitive to variation in thickness. Again, these plots show that for constant K_1/K_o ratios less than 1, the initially stiffer subgrade leads to the greater increase in stress and deflection. This suggests that if there is the possibility for the development of soft spots in the subgrade material, it may be better to avoid the use of stiff subgrades.

In Figure 13, there is a clear indication of what may happen, when, in addition to the reduction in subgrade reaction, 50% of load transfer is lost. Deflections are substantially increased near the point of discontinuity, especially when the load is over the weakened region. For instance, in Figure 13, for $K_1/K_o = 0.5$, the deflection amplification for a point under the moving load is 5.85 compared to 1.26 in Figure 9b. Another important aspect to note is the large relative movement which occurs between the edges of the slab (i.e., edges situated at the mid-point of the weakened region). This movement not only causes bumpy driving, but may also lead to further pavement distress.

As for stresses, Figure 13 demonstrates that for the special case considered, the maximum stress need not occur under the moving load. In fact, the maximum stress experienced in this case is not only behind the load, but is also correspondingly higher than that obtained in Figure 9b. For the other two positions of the load, the use of no-moment-transfer and 50% shear transfer did not greatly influence the values previously obtained in Figure 9 for maximum stress, as long as the ratio of K_1/K_o was high (approximately = 0.8); however, when the ratio of K_1/K_o was lowered to 0.25, significant reduction in stress was experienced. Obviously, at high values of K_1/K_o, the load in this case has to be fairly close to the weakened region before conditions existing at the mid-point of the region become important.

SUMMARY AND CONCLUSIONS

In part I of this study it is shown that when a pavement is subjected to upward curling, it is possible to experience an increase in maximum positive deflection as velocity is increased and then a decrease (in maximum positive deflection) with further increases of velocity. The velocity at which the decrease in maximum positive deflection sets in seems to depend on several factors which include the thickness of the slab, the temperature difference between its surfaces, the stiffness of the foundation and the degree of damping. In general, it appears that higher velocities are needed to effect a decrease in maximum positive deflection as slab thickness, stiffness of foundation and temperature increase, and as the degree of damping decreases.

In the case of stresses, maximum stress increased with increasing velocity. The thinner the pavement and the lower the value of subgrade reaction and damping coefficient, the higher the stress; however, there

was not a great difference in the values of stress obtained. What seemed to matter most, was the difference in temperature between the surfaces of the slab. Increases in the temperature difference resulted in large increases in stress and maximum positive deflection, for all values of velocity studied. This observation tends to indicate temperature difference between slab surfaces as the over-riding factor governing the magnitude of stress and deflection which may be obtained, while factors such as velocity, thickness of pavement, modulus of subgrade reaction and degree of damping act more or less to exaggerate or minimize the pavement distress caused by temperature and/or moisture gradients.

As for Part II, where the influence of a reduction in subgrade reaction was studied, it was clearly shown that both maximum deflection and maximum stress increased as the region became weaker. The increase experienced was quite pronounced when the load was within the weakened region and moving towards its center. Thus, if a pavement is designed with a particular value of subgrade reaction, and a weakened zone develops due to the softening of ground (as may be the case during Spring thaw) the stresses produced in the slab in the near vicinity of the weakened zone will very likely be higher than anticipated.

If there is significant reduction of subgrade support (even to within 75% of the original value of K_o), maximum deflection and stress should still follow the well known trend and decrease with increasing velocity; however, when there is complete loss of subgrade support, both the maximum stress and deflection can increase with increasing velocity. For the hypothetical case where pavements of equal thickness are built on different subgrade material, it appears that for a constant percentage loss of subgrade support, the pavement built on the stiffer subgrade material should

experience a greater increase in stress and deflection. In the case where there is also a loss of load transfer, deflections as well as stresses may be substantially increased and large relative movement may be experienced near the points of discontinuity in the surface of the slab.

In conclusion, we find that on the basis of the assumptions stated herein, this study indicates that: (1) Contrary to common opinion, an increase in velocity can produce an increase in deflection and stress, (2) Of all the variables considered herein (temperature differences, velocities, thickness of pavement, moduli of subgrade reaction and degrees of damping), the temperature difference between slab surfaces was shown to be the over-riding factor governing the magnitude of maximum stress and deflection, (3) There is reason to believe that any meaningful interpretation of the performance of pavements as determined by measured strains and/or deflections of the slab, must give due regard to the effects of warping, (4) Stresses and deflections within a slab can be sensitive to localized reductions in subgrade support.

ACKNOWLEDGMENTS

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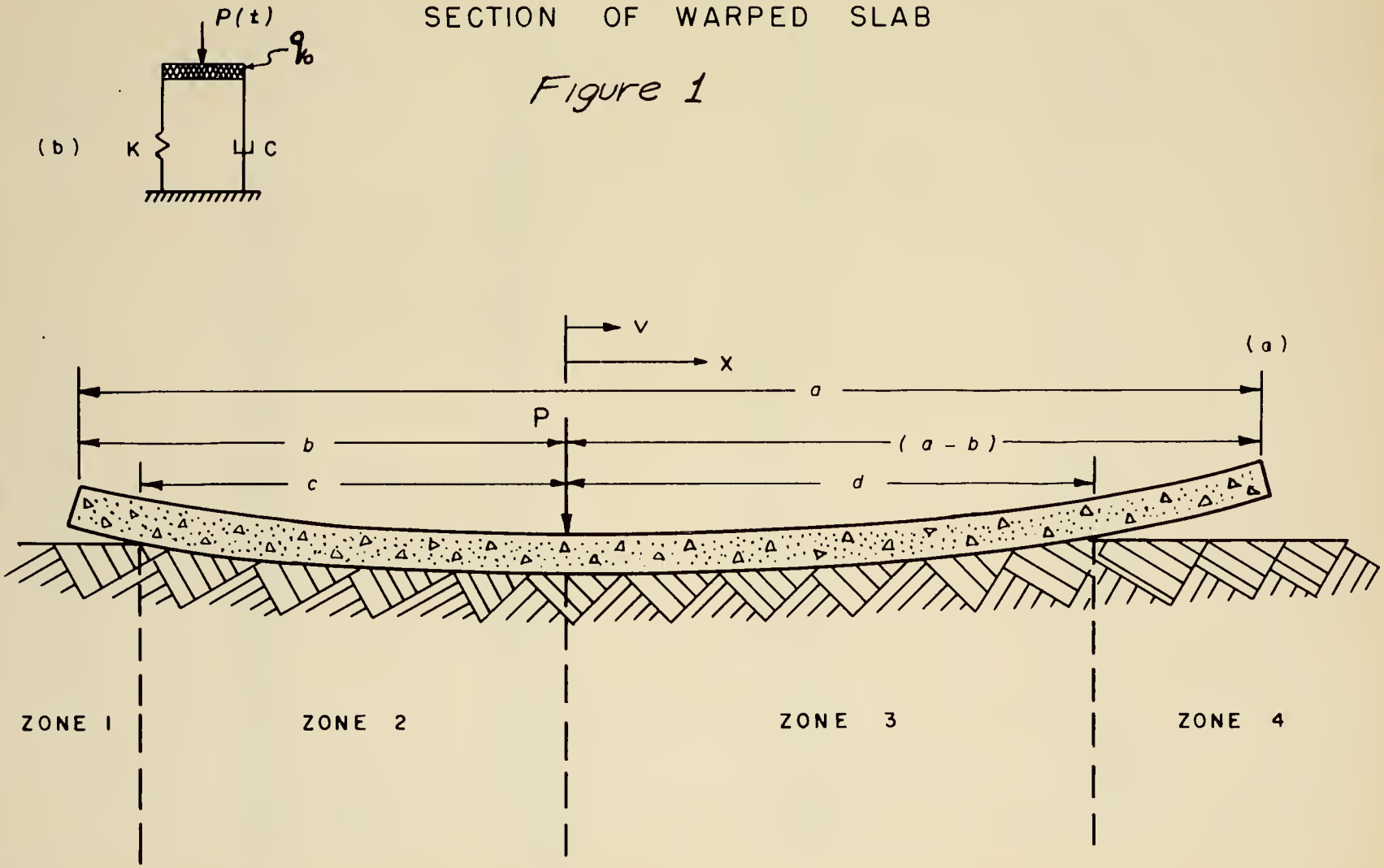

SECTION OF WARPED SLAB

Figure 1

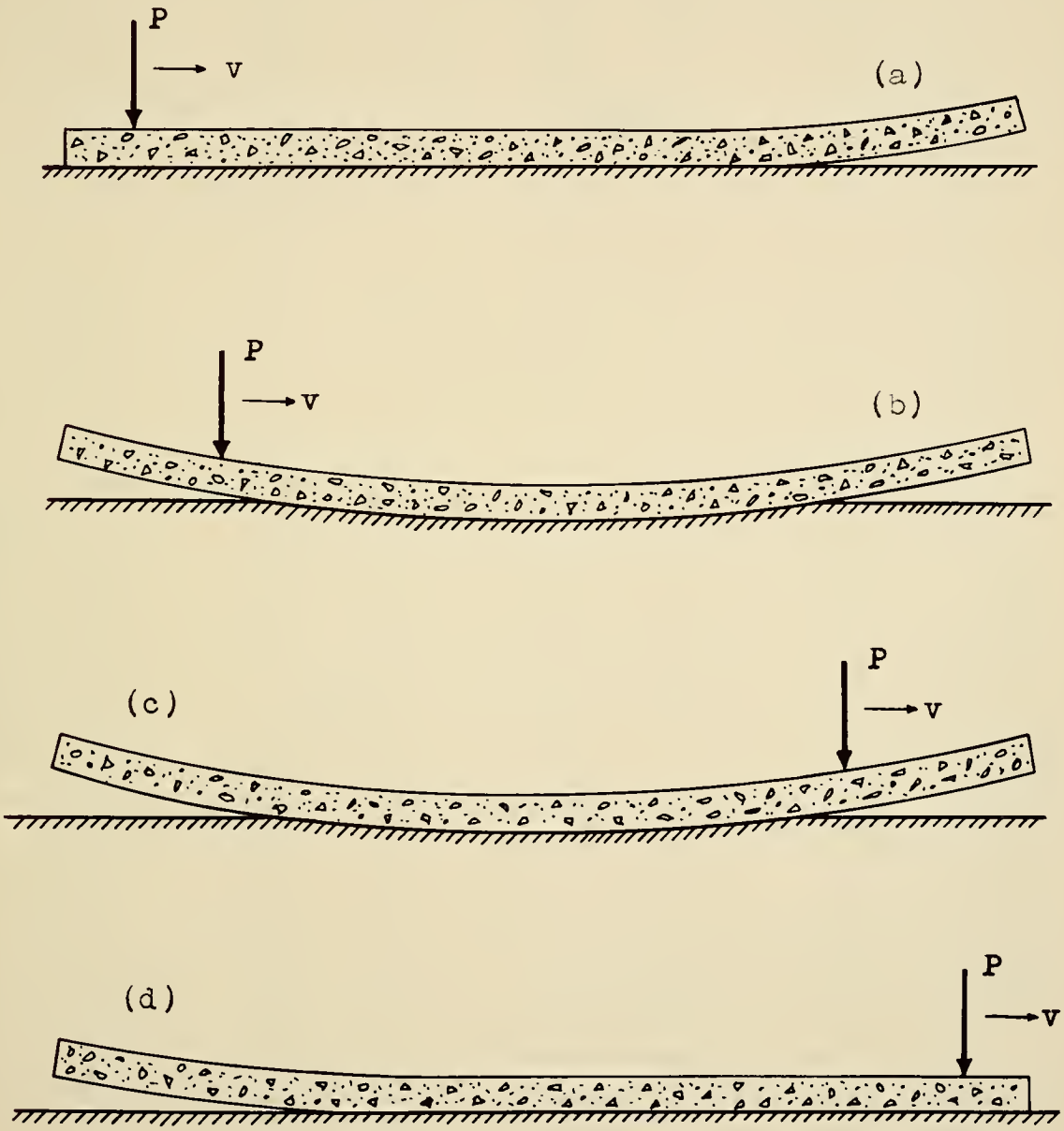

Figure 2. Typical Deflection Patterns

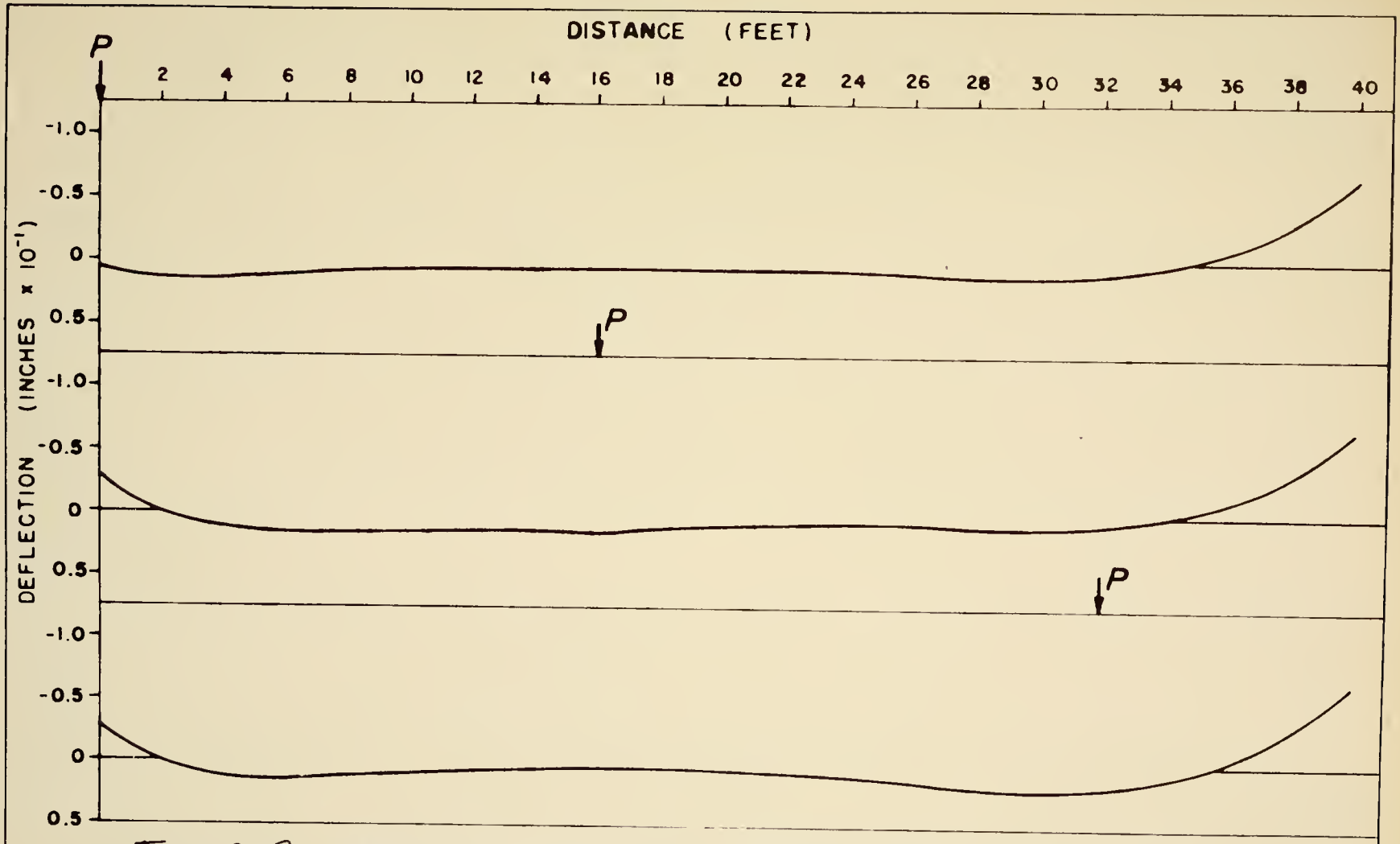

Figure 3a. Deflection versus Position of Moving Load (K_o= 100 pci, H=8 ins. C_r=1.5, ΔT=30 Deg., v=40 mph.)

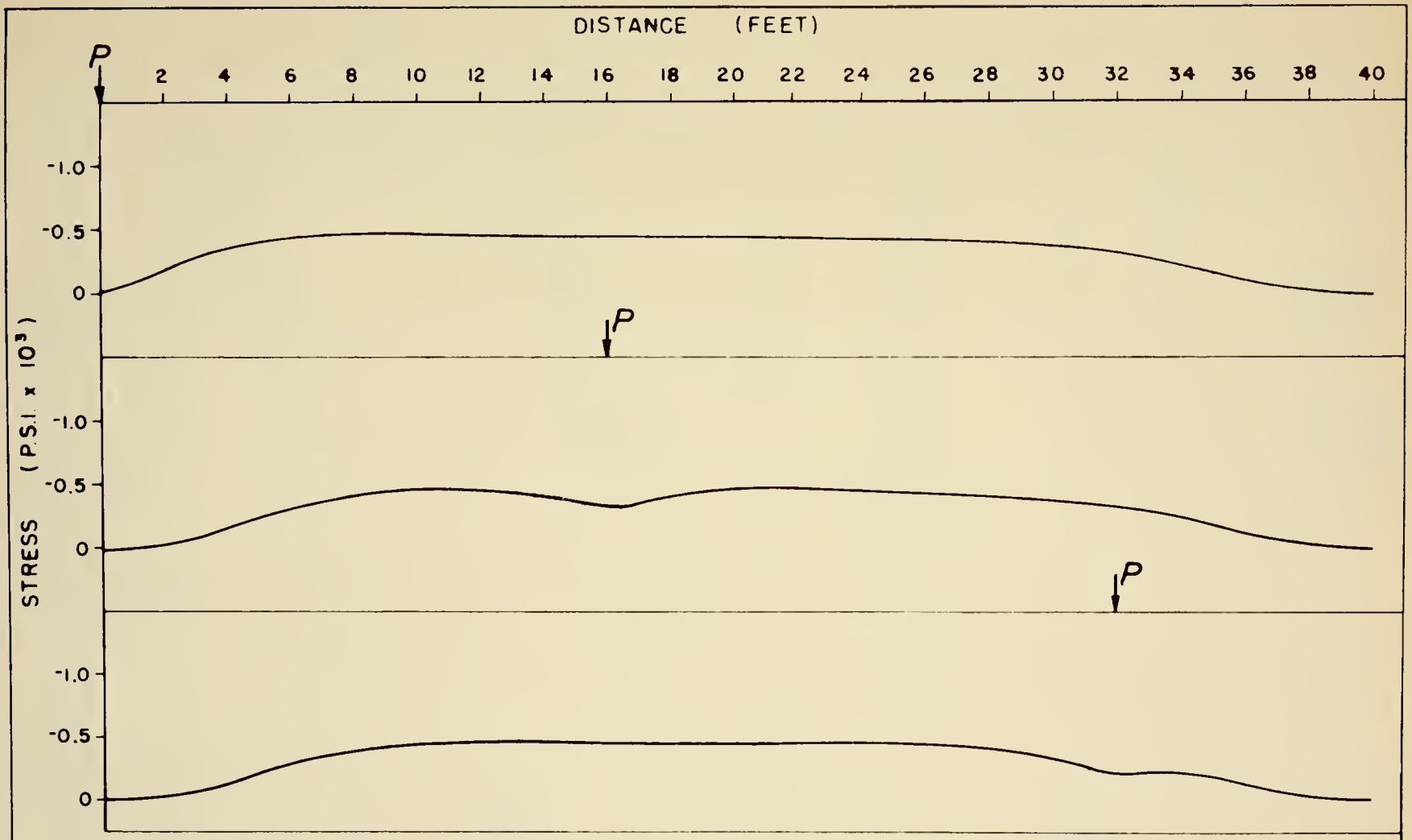

Figure 3b. Tensile Stress versus Position of Moving Load (K_o= 100 pci., H=8 ins., C_r=1.5, ΔT= 30 Deg., v=40 mph.)

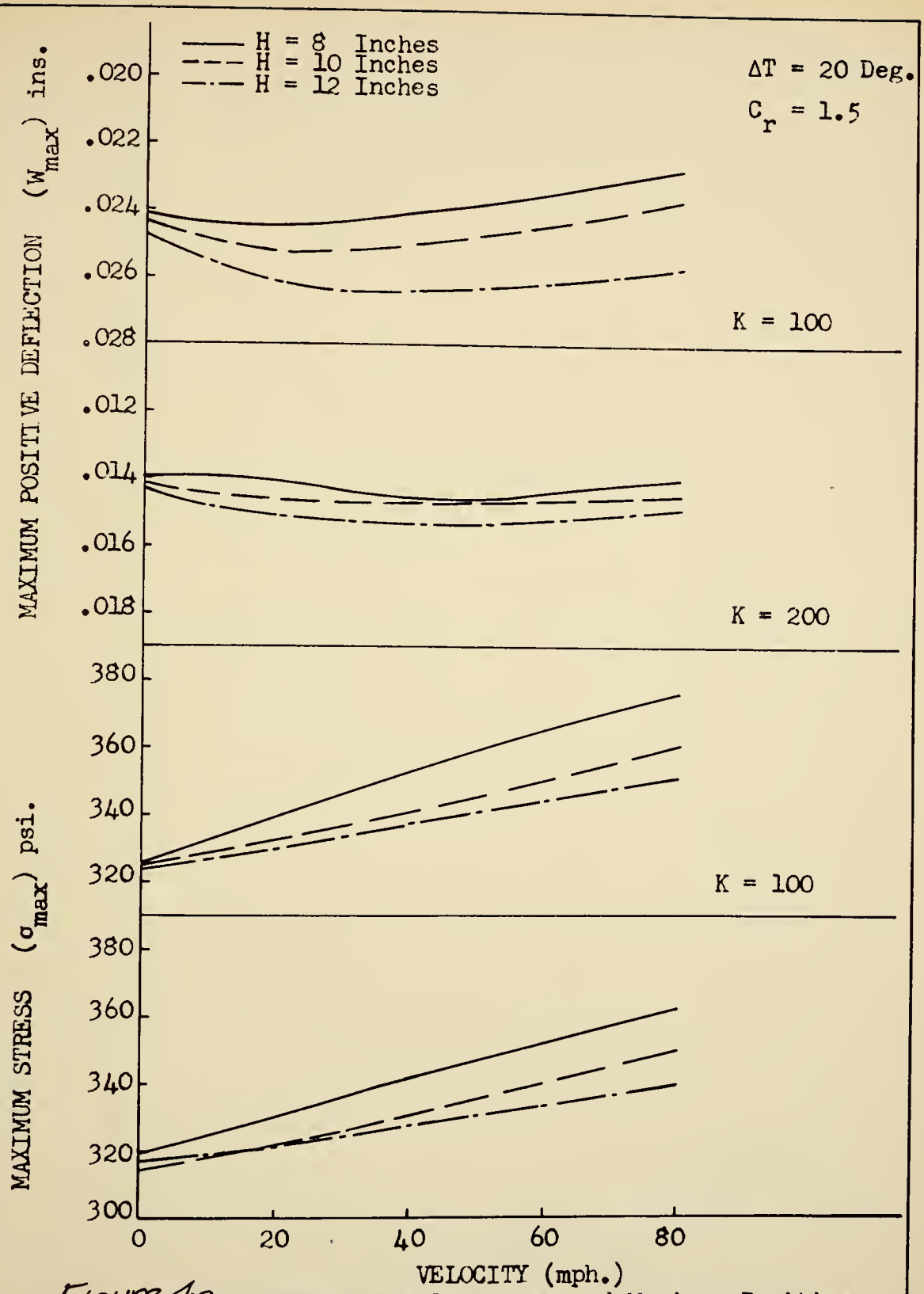

Figure 4a. Maximum Tensile Stress and Maximum Positive Deflection versus Velocity of Moving Load (C_r=1.5, ΔT=20 Deg.)

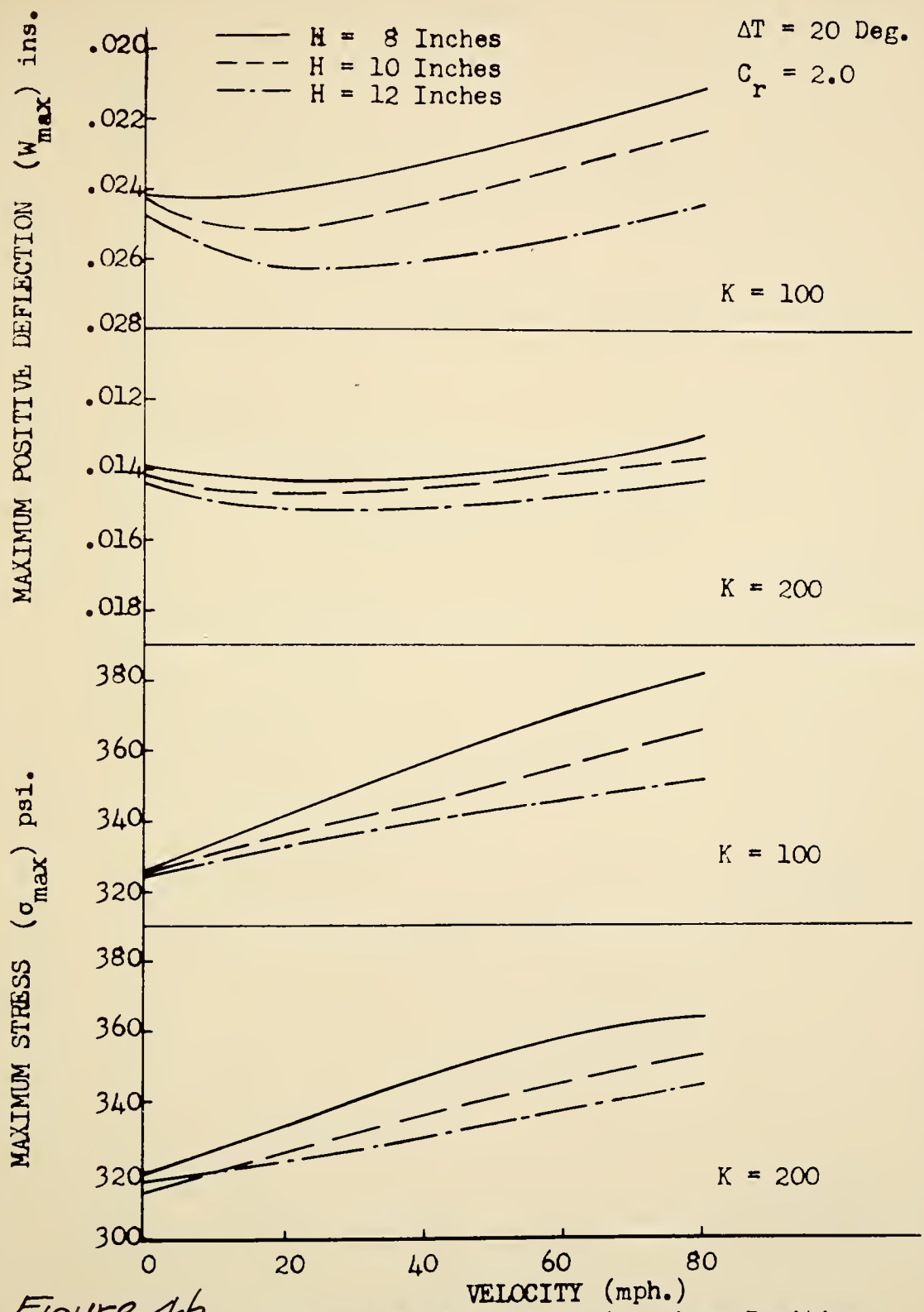

Figure 4b. Maximum Tensile Stress and Maximum Positive Deflection versus Velocity of Moving Load (C_r=2.0, ΔT=20 Deg.)

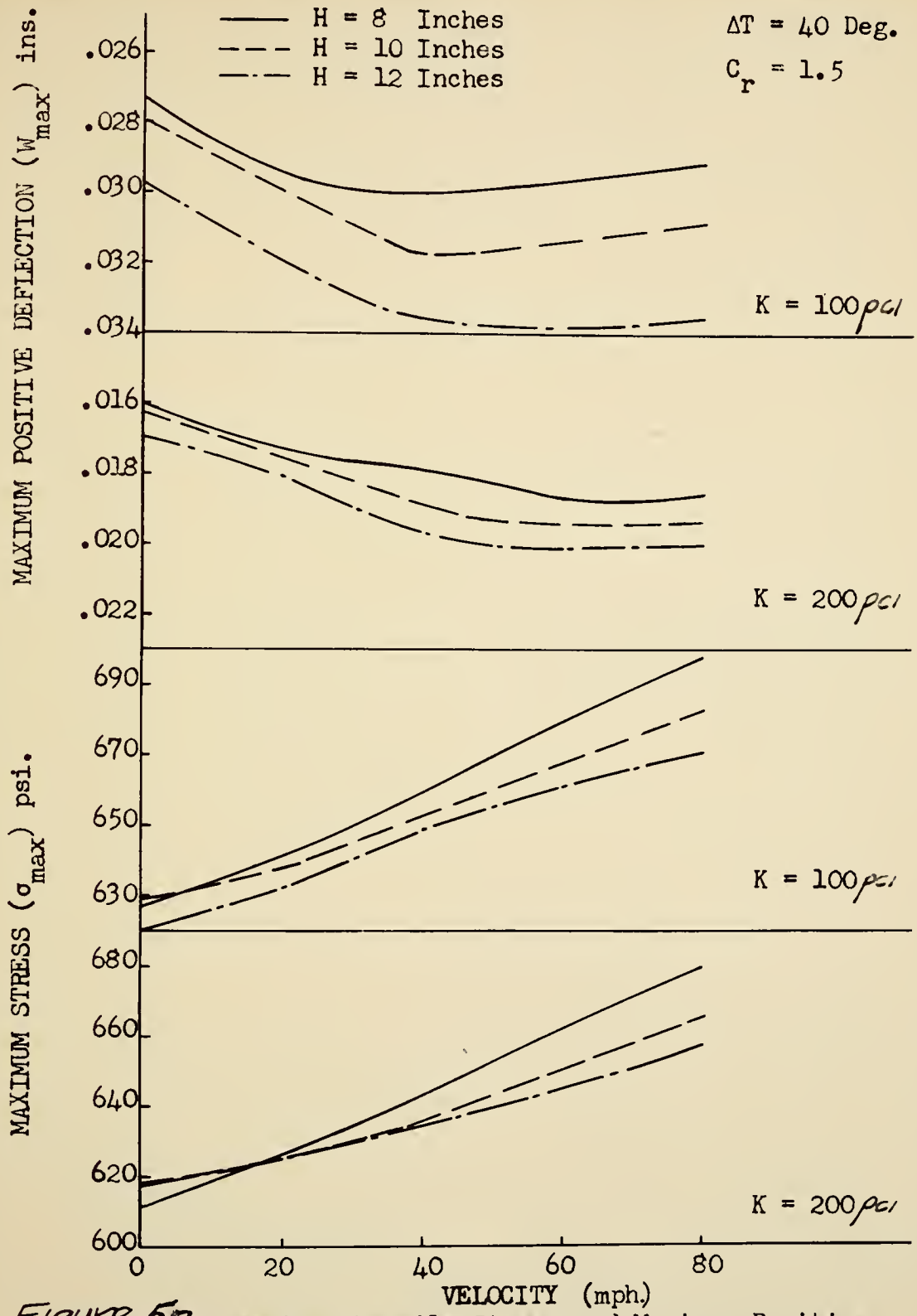

Figure 5a. Maximum Tensile Stress and Maximum Positive Deflection versus Velocity of Moving Load (C_r=1.5, ΔT =40 Deg.)

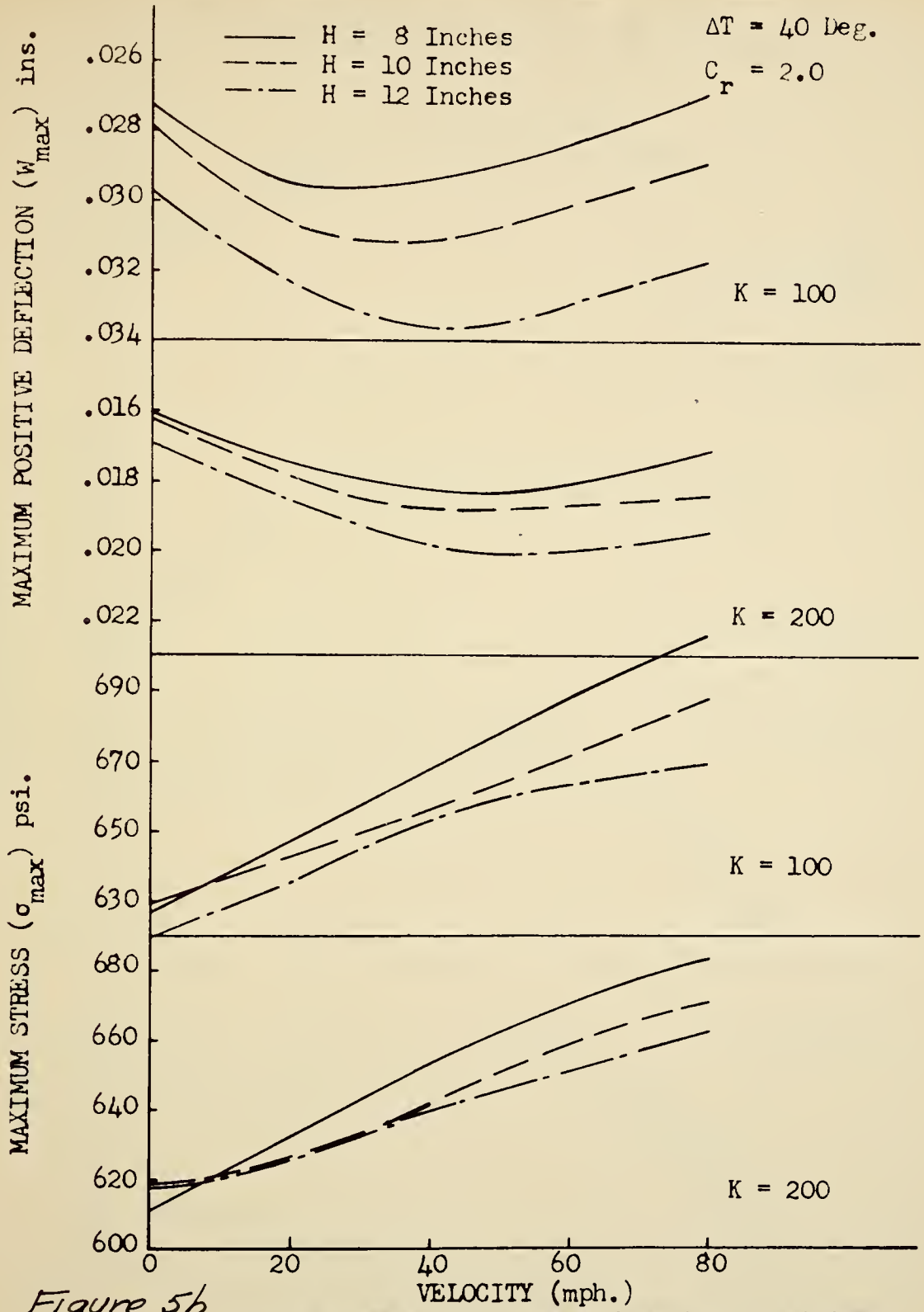

Figure 5b.

Maximum Tensile Stress and Maximum Positive Deflection versus Velocity of Moving Load (C_r=2.0, ΔT=40 Deg.)

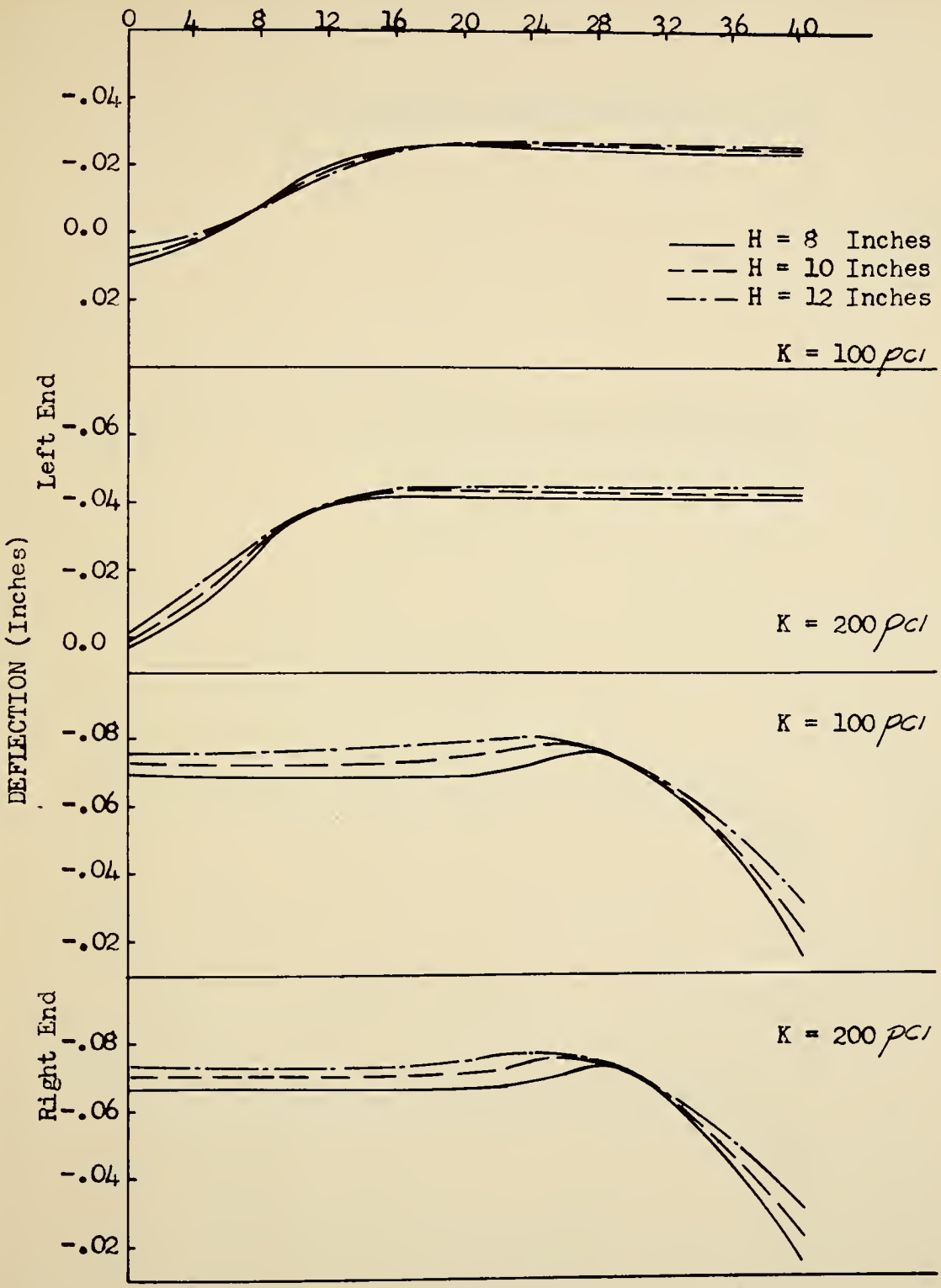

Figure 6 Deflection at Ends of Slab versus Position of Moving Load (C_r=1.5, ΔT=30 Deg., v=40 mph.)

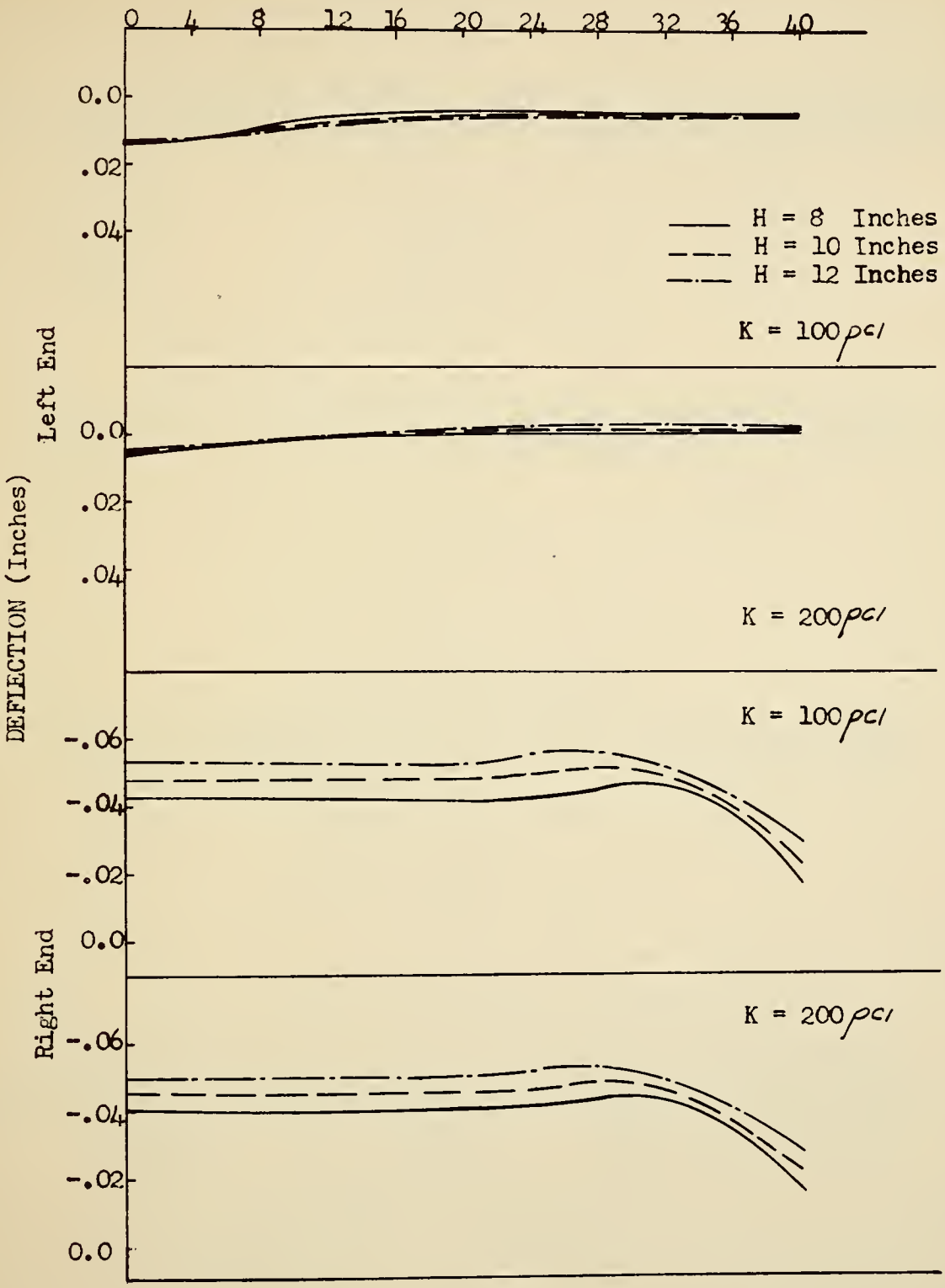

Figure 7 Deflection at Ends of Slab versus Position of Moving Load (C_r=1.5, ΔT=30 Deg., v=80 mph.)

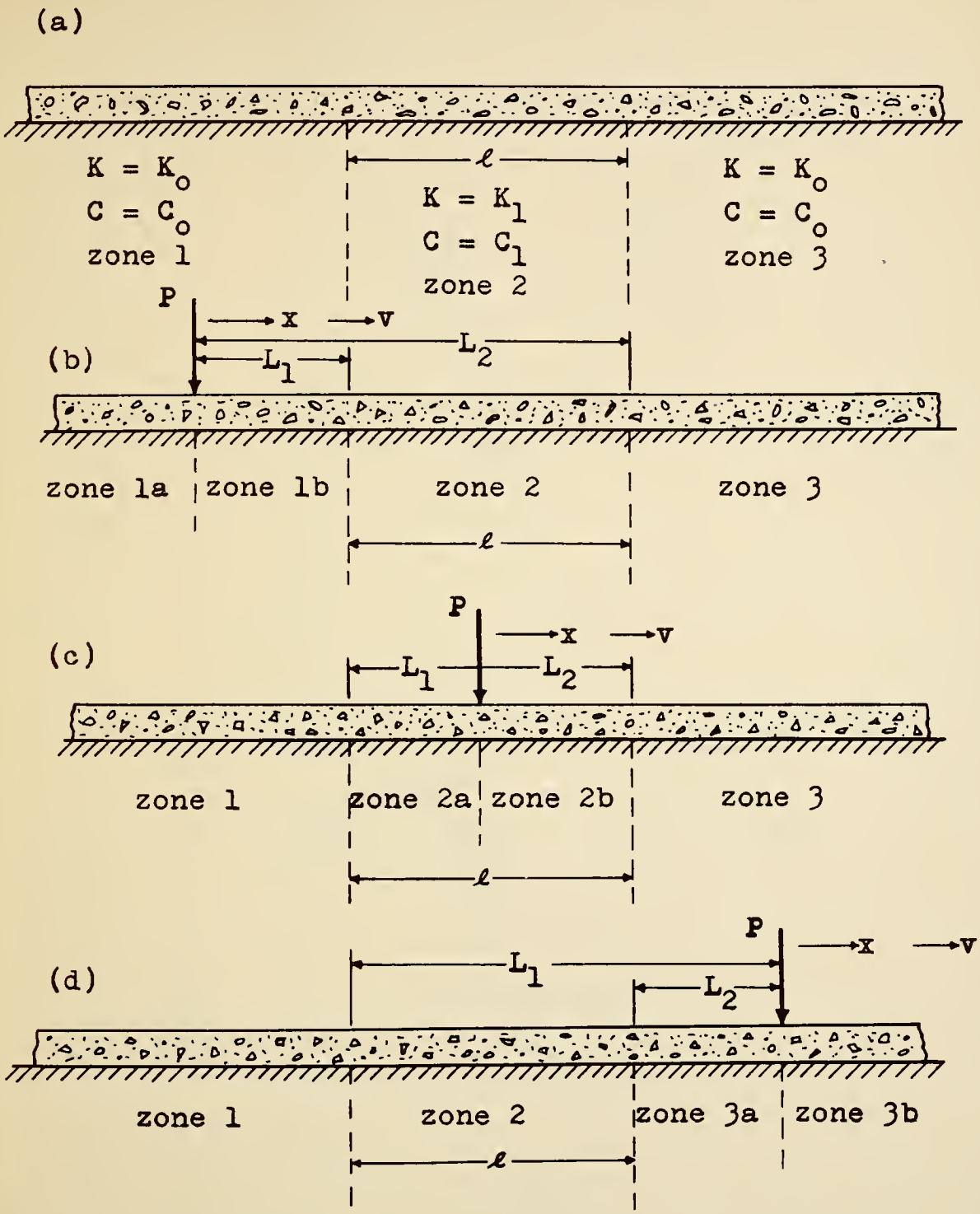

Figure 8. **Infinite Slab Over Region of Reduced Subgrade Reaction**

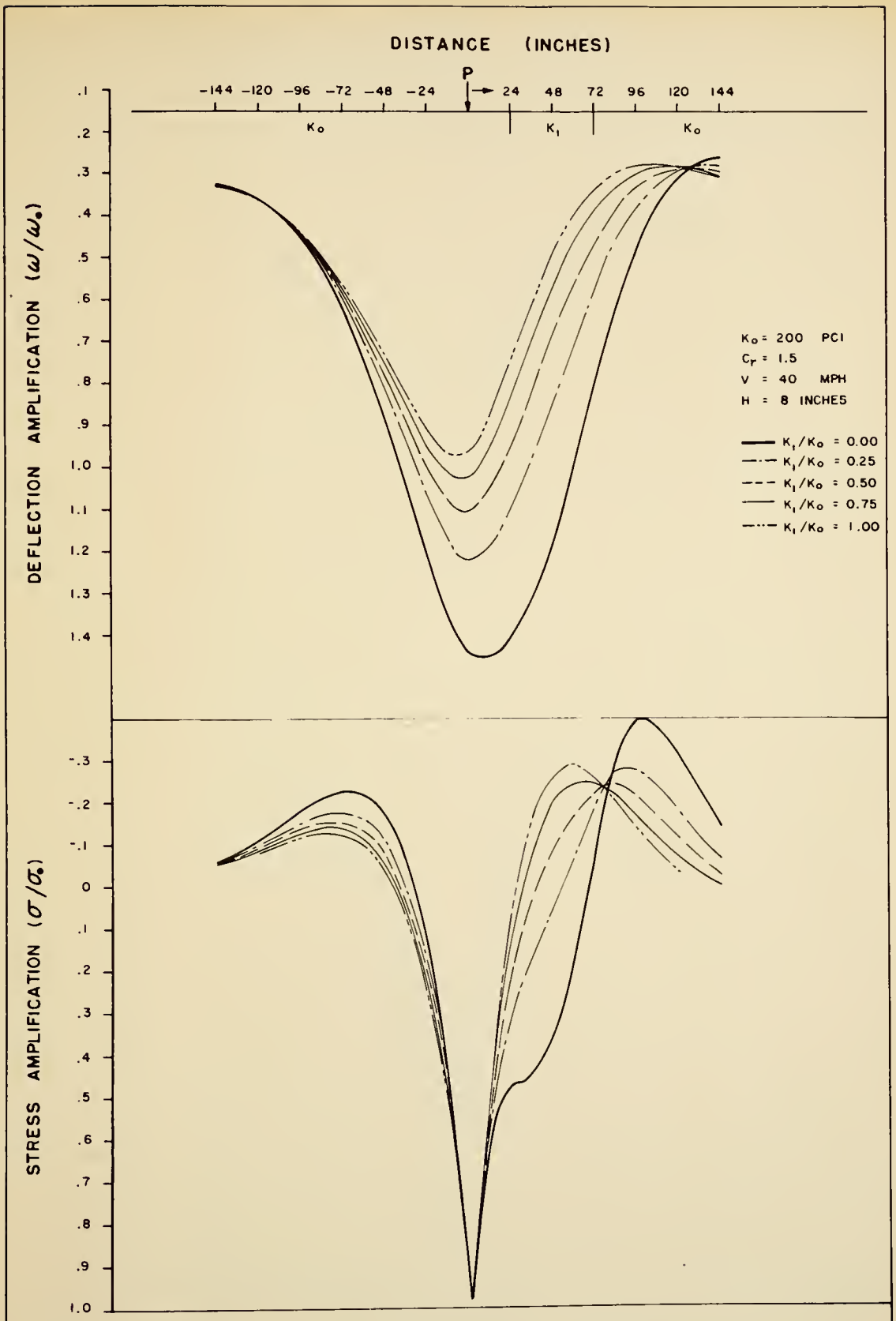

Figure 9a Deflection and Stress Amplification for Load 2 Ft. Behind Region of Reduced Subgrade Reaction

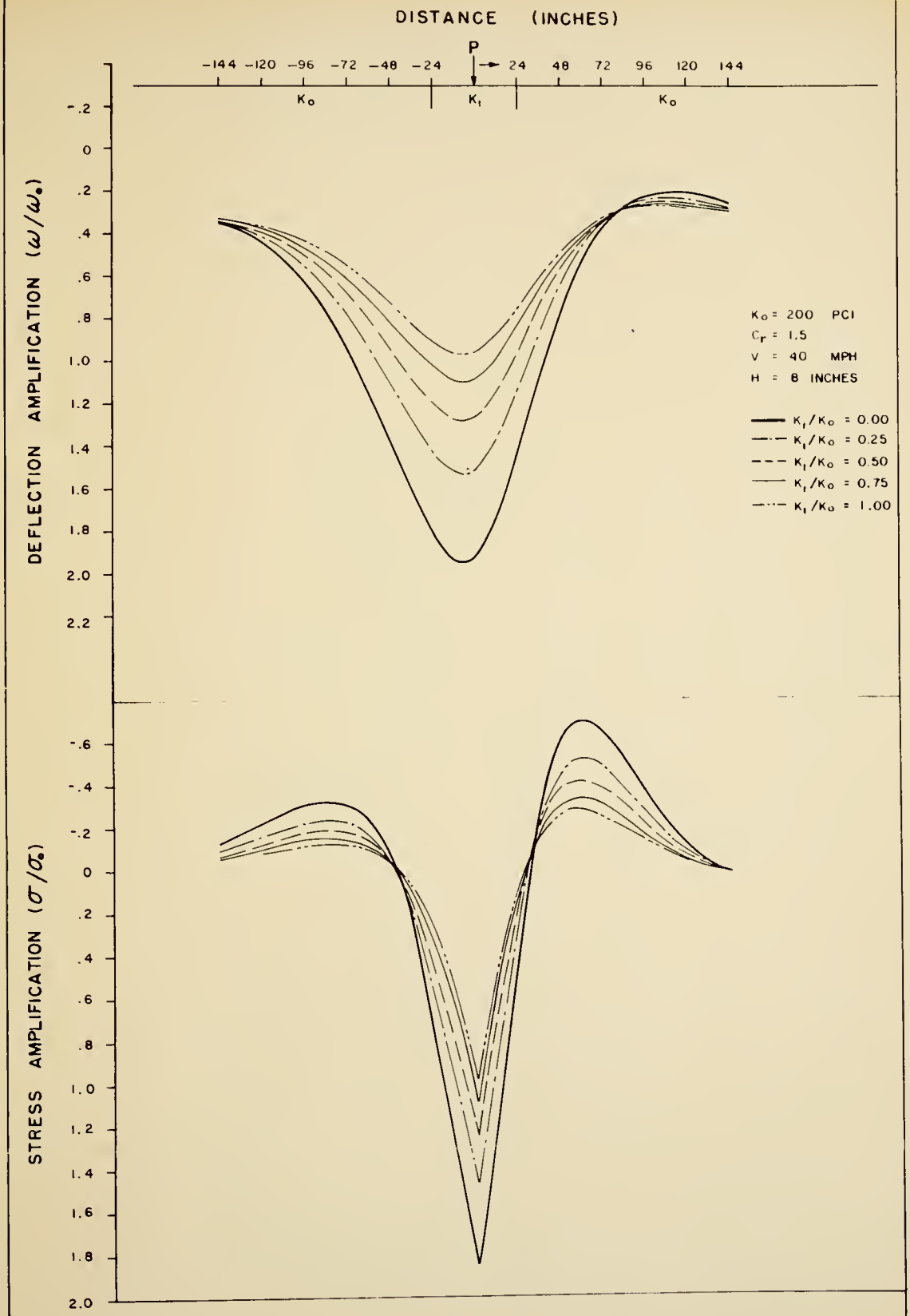

Figure 9b

Deflection and Stress Amplification for Load at Mid-point of Region of Reduced Subgrade Reaction

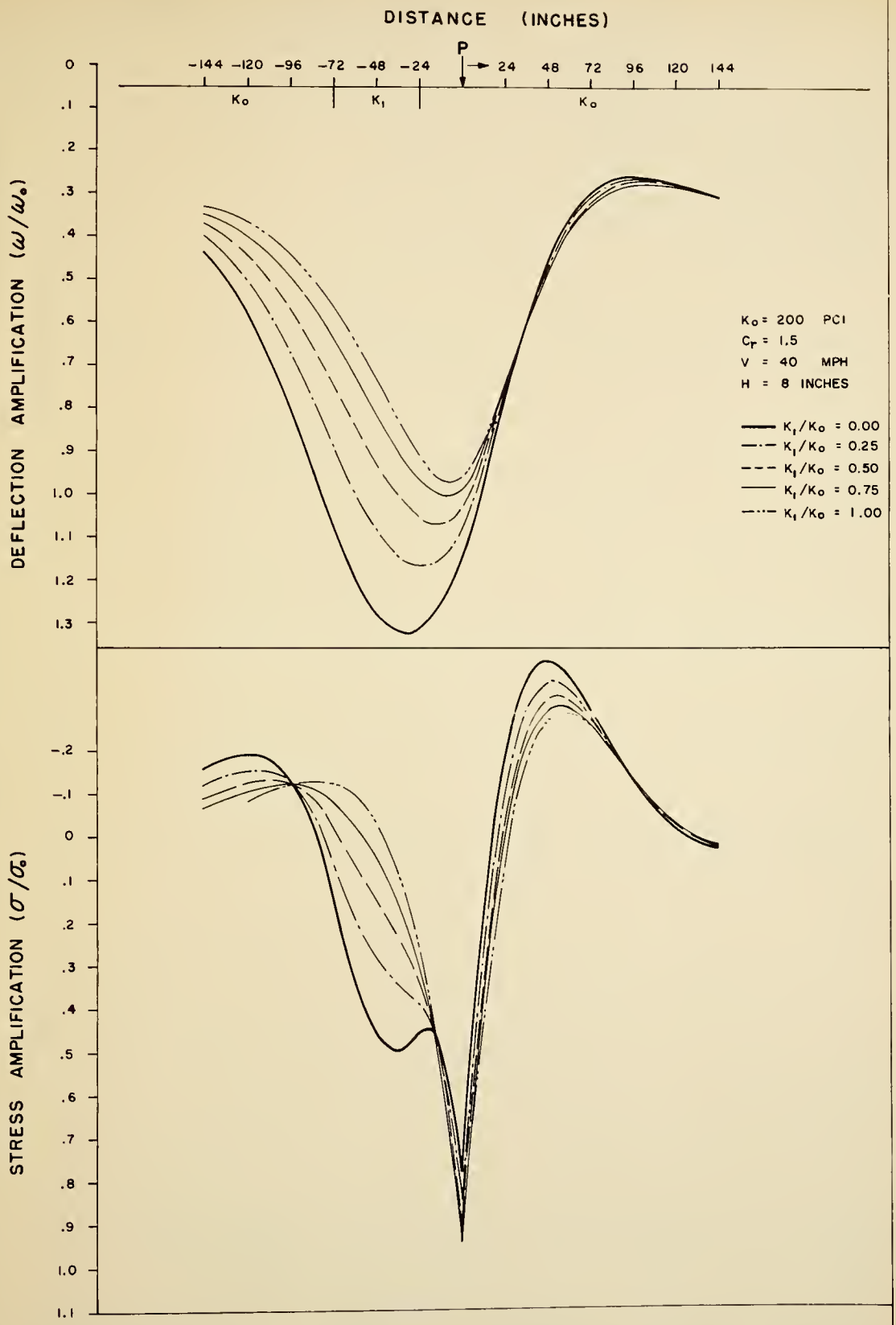

Figure 9c Deflection and Stress Amplification for Load 2 Ft. Ahead of Region of Reduced Subgrade Reaction

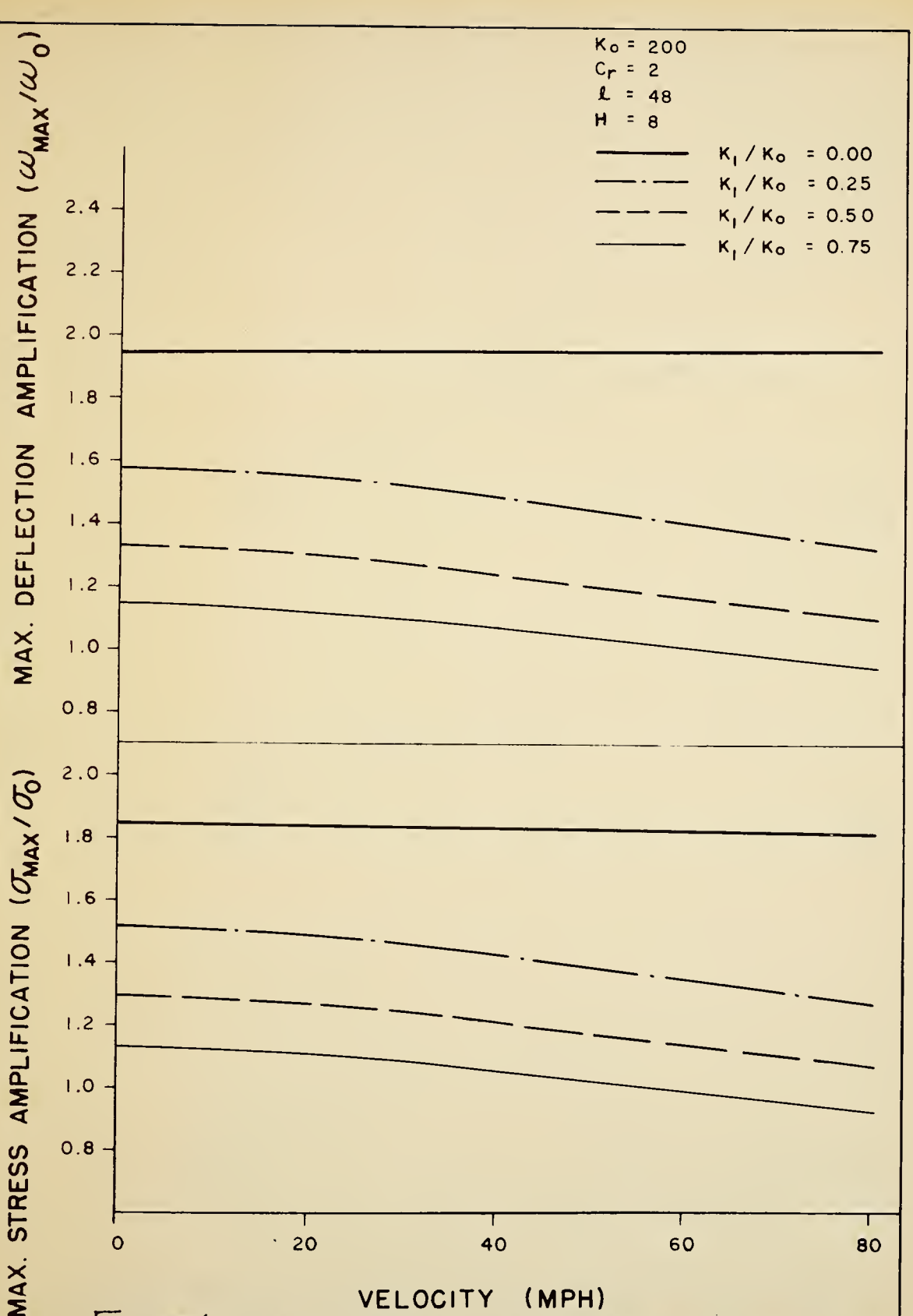

Figure 10. Maximum Deflection and Stress Amplification vrs Velocity (K_o=200 pci., C_r=2.0)

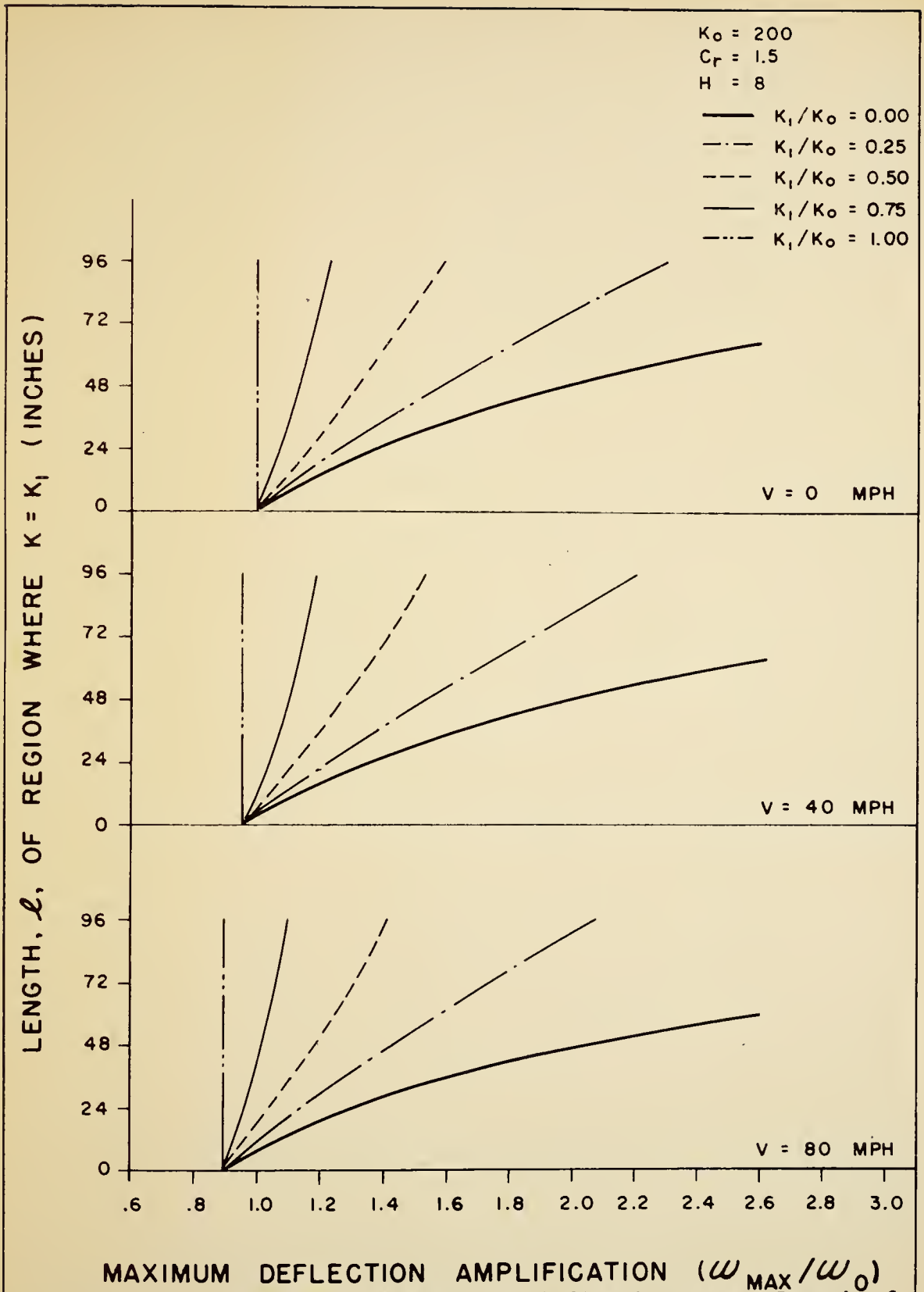

Figure 11a. Maximum Deflection Amplification versus Length of Region of Reduced Subgrade Reaction (K_o=200 pci., C_r=1.5)

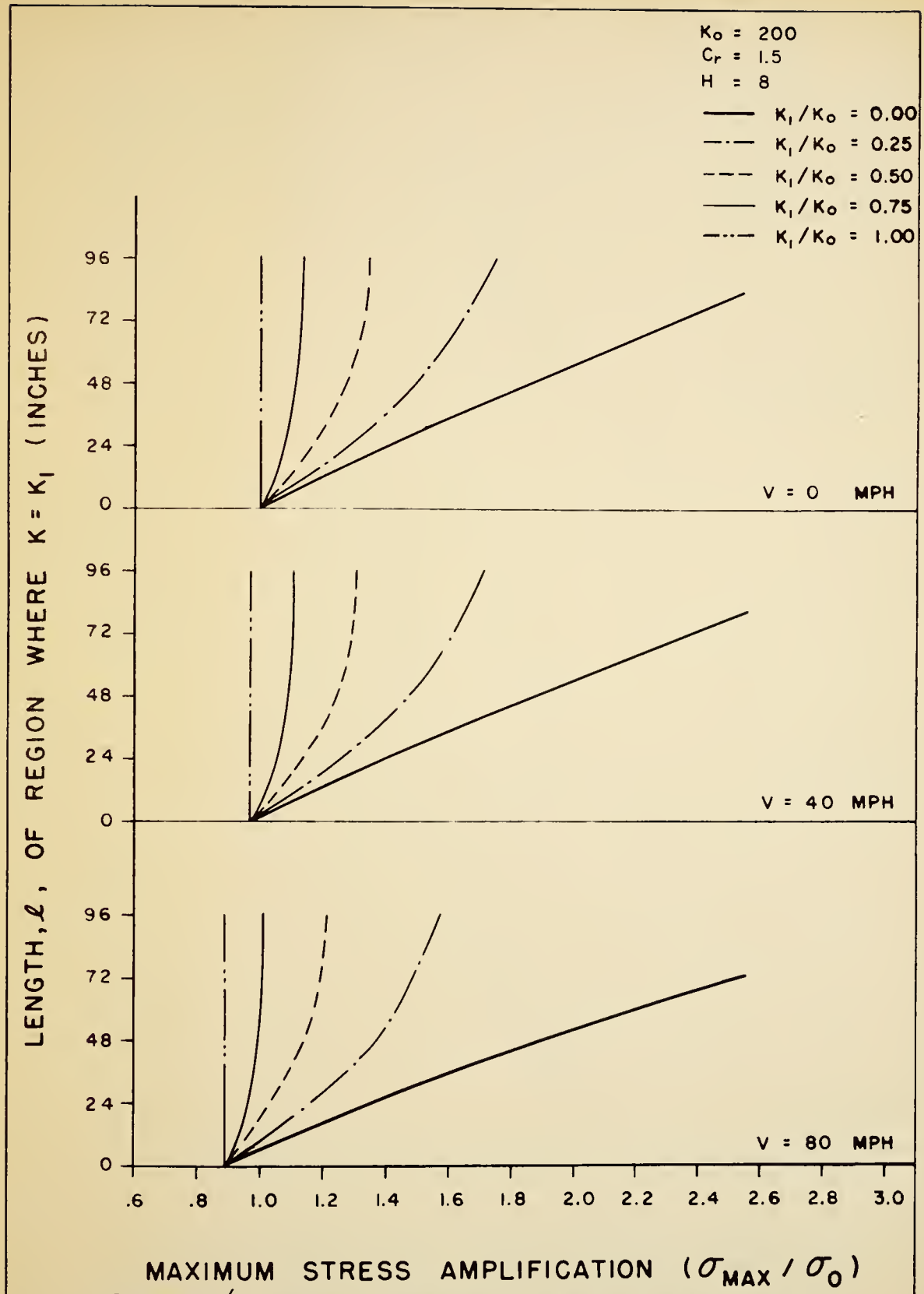

Figure 11b Maximum Stress Amplification versus Length of Region of Reduced Subgrade Reaction (K_o=200 pci., C_r=1.5)

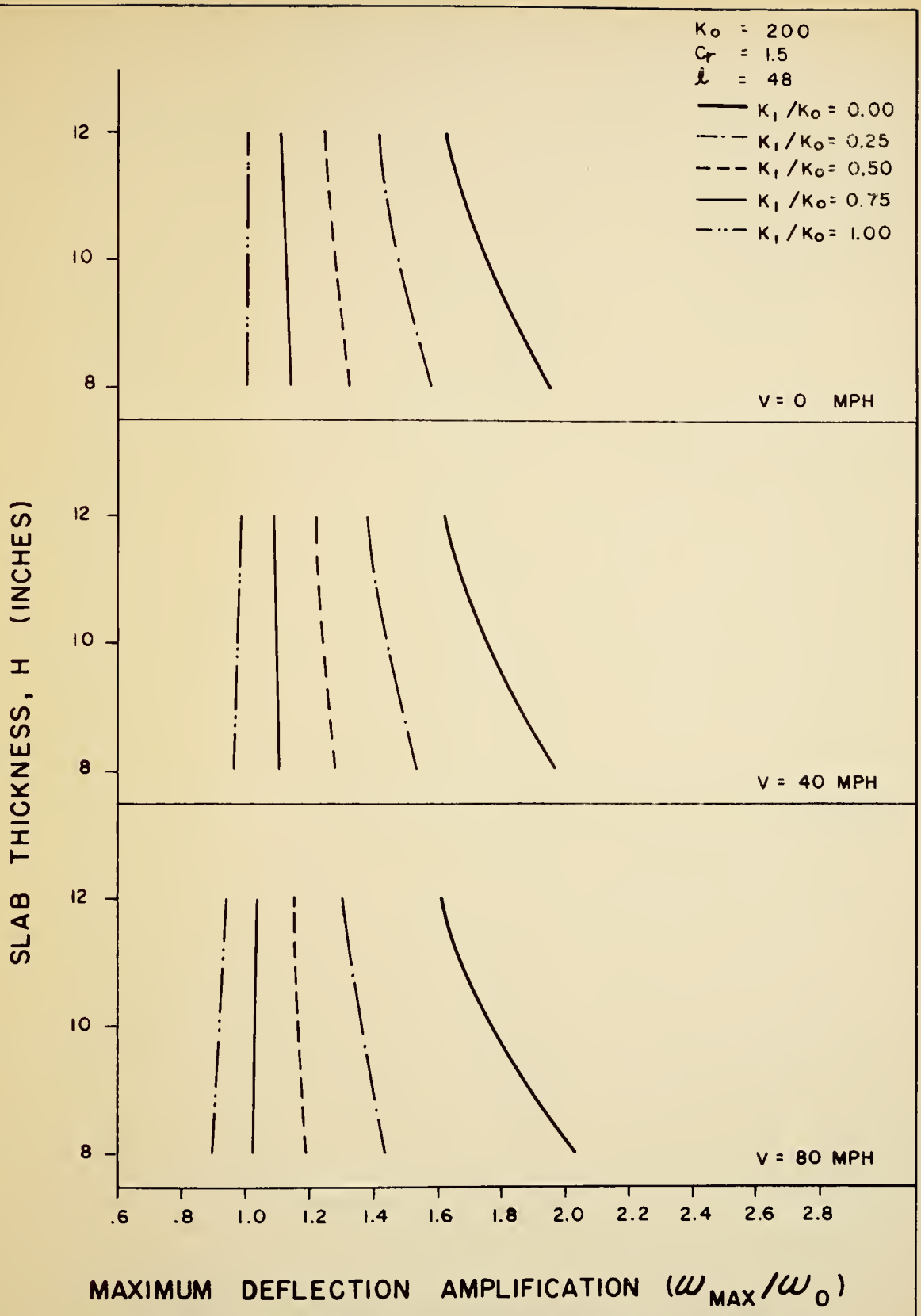

Figure 12a. Maximum Deflection Amplification versus Slab Thickness (K_o=200 pci., C_r=1.5)

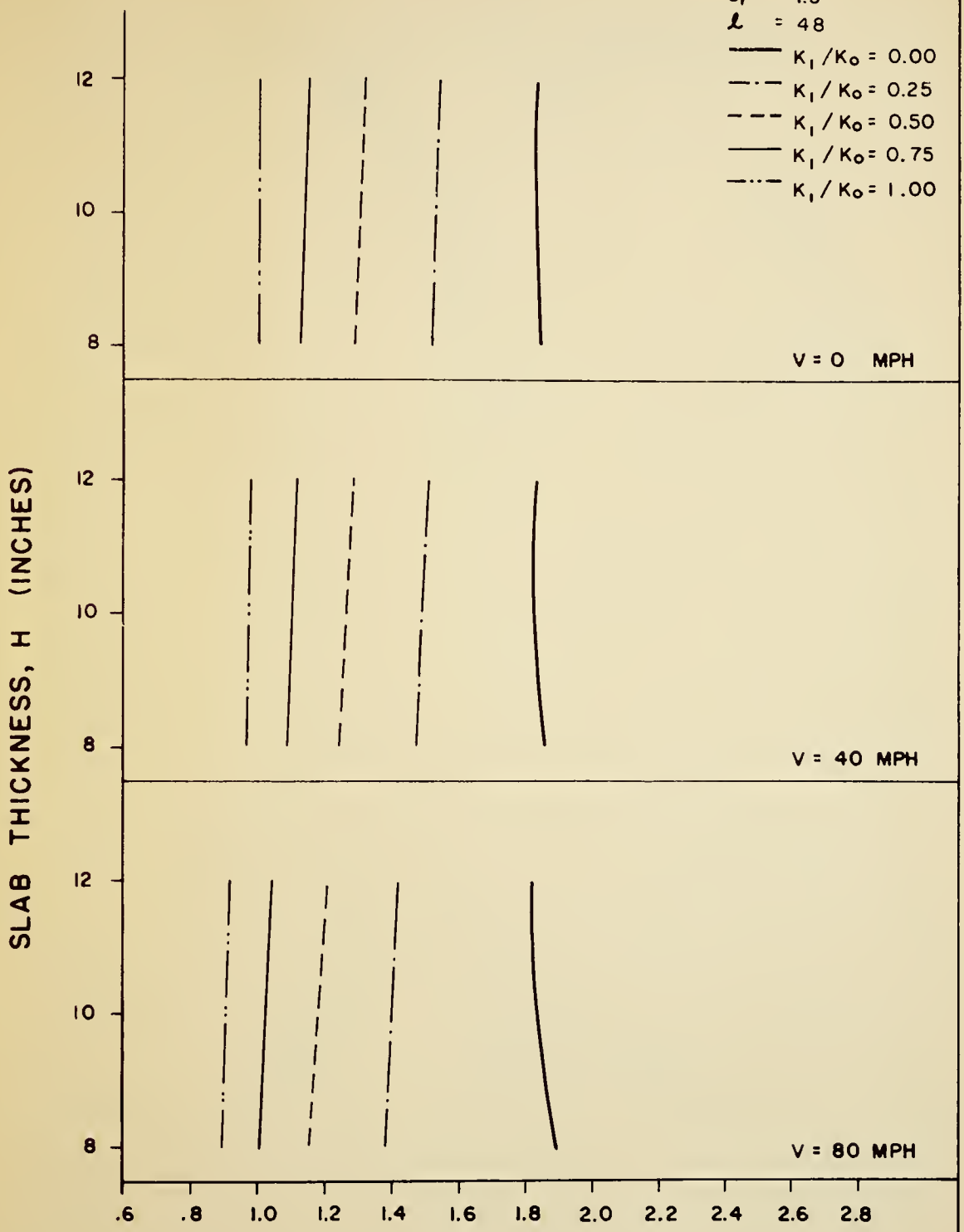

Figure 12b. Maximum Stress Amplification versus Slab Thickness (K_o=200 pci., C_r=1.5)

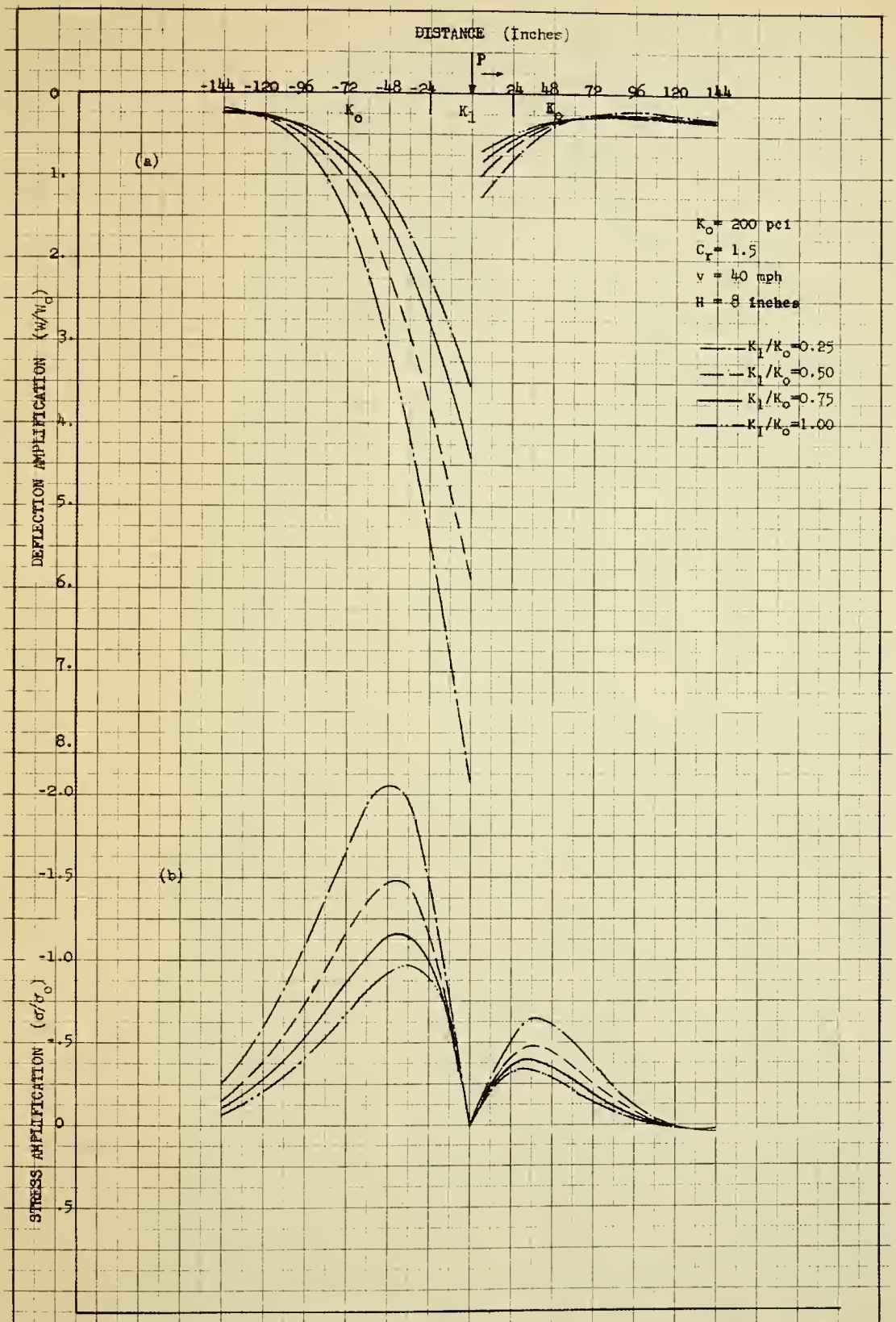

Figure 13. Deflection and Stress Amplification for Load at Mid-point of Region of Reduced Subgrade Reaction (No moment transfer across mid-point of Region)